Vacuum Boundary Problem of the Metric Produced by Perfect Fluid Mass Distributions

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1 Introduction

We consider the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}R_{\alpha\beta}) = \frac{8\pi G}{c^4}T_{\mu\nu}$$

(1.1)

for the energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (c^2 \rho + P)U^\mu U^\nu - Pg^{\mu\nu}.$$  

(1.2)

Here $R_{\mu\nu}$ is the Ricci tensor associated with the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

(1.3)

$G$ is the gravitational constant, $c$ the speed of light, $\rho$ the mass density, $P$ the pressure and $U^\mu$ is the 4-dimensional velocity.
We suppose that the pressure $P$ is a given function of the density $\rho$ and suppose the following

**Assumption 1.** $P$ is a smooth function of $\rho > 0$ such that $0 < P, 0 < dP/d\rho < c^2$ for $\rho > 0$, and $P \to 0$ as $\rho \to +0$. Moreover there are positive constants $A, \gamma$ and a smooth function $\Upsilon$ defined on $\mathbb{R}$ which is analytic on a neighborhood of 0 such that $\Upsilon(0) = 0$ and

$$P = A\rho^{\gamma}(1 + \Upsilon(A\rho^{\gamma^{-1}}/c^2)).$$

for $\rho > 0$. We assume that $1 < \gamma < 2$. 
We note that there are smooth functions $\Upsilon_u, \Upsilon_\rho, \Upsilon_P$ analytic on a neighborhood of 0 such that $\Upsilon_u(0) = \Upsilon_\rho(0) = \Upsilon_P(0) = 0$ and

\begin{align}
  u &= \frac{\gamma A}{\gamma - 1} \rho^{\gamma - 1} (1 + \Upsilon_u(A\rho^{\gamma - 1}/c^2)), \\
  \rho &= A_1 u^{\frac{1}{\gamma - 1}} (1 + \Upsilon_\rho(u/c^2)), \\
  P &= AA_1^{\gamma} u^{\frac{\gamma}{\gamma - 1}} (1 + \Upsilon_P(u/c^2)).
\end{align}

Here $A_1 := \left(\frac{\gamma - 1}{\gamma A}\right)^{\frac{1}{\gamma - 1}}$. 


Example: Equation of state of neutron stars

\[ P = Kc^5 \int_0^z \frac{q^4 dq}{\sqrt{1 + q^2}} \]

\[ = \frac{3}{8} Kc^5 \left( 5(1 + z^2) \left( \frac{2}{3} z^2 - 1 \right) + \log(z + \sqrt{1 + z^2}) \right), \]

\[ \rho = 3Kc^3 \int_0^z \sqrt{1 + q^2} q^2 dq \]

\[ = \frac{3}{8} Kc^3 (2z^2 + 1) (z \sqrt{1 + z^2} - \log(z + \sqrt{1 + z^2})). \]

Then \( A = \frac{1}{5} K^{-2/3} \) and \( \gamma = \frac{5}{3} \).
2 Spherically symmetric problem

In this section, for the sake of generality, we consider the Einstein-de Sitter equation

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}R_{\alpha\beta}) - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]  

(2.1)

instead of the Einstein equation (1.1). Here \( \Lambda \) is the cosmological constant, which is supposed to be positive.
We consider spherically symmetric metrics of the form

\[ ds^2 = e^{2F(t,r)} c^2 dt^2 - e^{2H(t,r)} dr^2 - R(t, r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

We suppose that the system of coordinates is **co-moving**, that is,

\[ U^0 = e^{-F}, \quad U^1 = U^2 = U^3 = 0 \]

or

\[ U^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{ce^F} \frac{\partial}{\partial t} \]

for \( x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi. \)
The Einstein-de Sitter-Euler equations turn out to be

\begin{align}
  e^{-F} \frac{\partial R}{\partial t} &= V \quad (2.2a) \\
  e^{-F} \frac{\partial \rho}{\partial t} &= -(\rho + P/c^2) \left( \frac{V'}{R'} + \frac{2V}{R} \right) \quad (2.2b) \\
  e^{-F} \frac{\partial V}{\partial t} &= -GR \left( \frac{m}{R^3} + \frac{4\pi P}{c^2} \right) + \frac{c^2 \Lambda}{3} R + \\
  &\quad \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right) \frac{P'}{R'(\rho + P/c^2)} \quad (2.2c) \\
  e^{-F} \frac{\partial m}{\partial t} &= -\frac{4\pi}{c^2} R^2 PV \quad (2.2d)
\end{align}

Here $X'$ stands for $\partial X/\partial r$. 

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The coefficients of the metric are given by

\[ P' + F'(c^2 \rho + P) = 0 \]

and

\[ e^{2H} = \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-1} (R')^2. \]

In order to specify the function \( F \), we introduce the state variable \( u \) by

\[ u = \int_0^\rho \frac{dP}{\rho + P/c^2}. \]

Then there should exist a positive constant \( C \) such that

\[ e^F = Ce^{-u/c^2}. \]
To fix the idea we shall take $C = \sqrt{\kappa_+}$ with a positive constant $\kappa_+$ specified in the next Subsection. But the choice of the constant $C$ is free by change of the scale of $ct$. 
We put
\[ m = 4\pi \int_0^r \rho R^2 R' \, dr, \]
supposing that \( \rho \) is continuous at \( r = 0 \). The coordinate \( r \) can be changed to \( m \), supposing that \( \rho > 0 \), and the equations are reduced to
\begin{align}
 e^{-F} \left( \frac{\partial R}{\partial t} \right)_m &= \left( 1 + \frac{P}{c^2 \rho} \right) V, \tag{2.3a} \\
 e^{-F} \left( \frac{\partial V}{\partial t} \right)_m &= \frac{4\pi}{c^2} R^2 PV \frac{\partial V}{\partial m} - GR \left( \frac{m}{R^3} + \frac{4\pi P}{c^2} \right) + \frac{c^2 \Lambda}{3} R + \\
 & \quad - \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right) \left( 1 + \frac{P}{c^2 \rho} \right)^{-1} 4\pi R^2 \frac{\partial R}{\partial m}. \tag{2.3b}
\end{align}
Here \( (\partial/\partial t)_m \) means the differentiation with respect to \( t \) keeping \( m \) constant. We will change the coordinate \( m \) to \( r \) later through a fixed equilibrium, and we shall construct solutions near the equilibrium.
2.1 Equilibrium

Let us consider solutions independent of $t$, that is, $F = F(r), H = H(r), \rho = \rho(r), V \equiv 0, R \equiv r$. The equations are reduced to the Tolman-Oppenheimer-Volkoff-de Sitter equation

\[
\frac{dm}{dr} = 4\pi r^2 \rho, \quad (2.4a)
\]

\[
\frac{dP}{dr} = -\left( \rho + \frac{P}{c^2} \right) \frac{G \left( m + \frac{4\pi r^3}{c^2} P \right) - \frac{c^2 \Lambda}{3} r^3}{r^2 \left( 1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} \frac{r^2}{r^2} \right)}. \quad (2.4b)
\]
For arbitrary positive central density $\rho_0$ there exists a unique solution germ $(m(r), P(r)), 0 < r \ll 1$, such that

\[
m = \frac{4\pi}{3} \rho_0 r^3 (1 + O(r^2)), \tag{2.5a}
\]

\[
P = P_0 - (\rho_0 + P_0/c^2) \left( \frac{4\pi G}{3} (\rho_0 + 3P_0/c^2) - \frac{c^2 \Lambda}{3} \right) \frac{r^2}{2} + O(r^4). \tag{2.5b}
\]

We denote

\[
\kappa(r,m) := 1 - \frac{2Gm}{c^2r} - \frac{\Lambda}{3} r^2, \tag{2.6}
\]

\[
Q(r,m,P) := G \left( m + \frac{4\pi r^3}{c^2} P \right) - \frac{c^2 \Lambda}{3} r^3. \tag{2.7}
\]

We concentrate ourselves to solutions satisfying $\kappa(r,m(r)) > 0$. 

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Let us use the concept of ‘monotone short solutions’ defined as follows:

**Definition 1.** A solution \((m(r), P(r))\), \(0 < r < r_+\), of (2.4a)(2.4b) is said to be **monotone-short** if \(r_+ < \infty\), \(dP/dr < 0\) for \(0 < r < r_+\), that is, \(Q(r, m(r), P(r)) > 0\), and \(P \to 0\) as \(r \to r_+ - 0\) and if both

\[
\kappa_+ := \lim_{r \to r_+ - 0} \kappa(r, m(r)) = 1 - \frac{2Gm_+}{c^2r_+} - \frac{\Lambda}{3}r_+^2 \tag{2.8}
\]

and

\[
Q_+ := \lim_{r \to r_+ - 0} Q(r, m(r), P(r)) = Gm_+ - \frac{c^2\Lambda}{3}r_+^3 \tag{2.9}
\]

are positive. Here

\[
m_+ := \lim_{r \to r_+ - 0} m(r) = 4\pi \int_0^{r_+} \rho(r)r^2dr. \tag{2.10}
\]
We suppose

**Assumption 2.** There is a monotone-short solution \((\bar{m}(r), \bar{P}(r))\), \(0 < r < r_+\), satisfying (2.5a)(2.5b).

We fix the equilibrium given by the monotone-short solution hereafter.
$u \sim \frac{Q}{r^2 K}$

\begin{align*}
S & \sim C \left( \frac{r - r_0}{r_0} \right)^{\frac{1}{\delta - 1}} \\
S & \sim \frac{1}{r_0^2 K_0} (r_0 - r) \\
I_0 & \leq I_1 \\
I_E(m, n) & \geq I
\end{align*}
As for sufficient conditions for the existence of monotone-short prolongations, see [TOVdS] *1 The sufficient conditions given by [TOVdS] Theorem 1 and [TOVdS] Theorem 2 assume relative smallness of Λ. The task to prove the existence of monotone-short solutions seems to be not so easy when Λ is permitted to be large. Actually, when Λ = 0, \( P(r) \) given by shooting from \( r = 0 \) remains monotone decreasing with respect to \( r \) automatically, and \( \kappa_+ \) turns out to be positive automatically if \( P(r) \) hits zero at \( r_+ < \infty \). [TOV] *2 proved that, when \( \Lambda = 0 \), it is sufficient for the existence of the finite zero \( r_+ \) that \( \gamma > \frac{4}{3} \) for \( \forall \rho_0 = \rho(O) > 0 \). But generally it is not the case when \( \Lambda > 0 \) is large.

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Anyway, the associated function $u = \bar{u}(r)$ turns out to be of the form

$$\bar{u}(r) = \frac{Q+}{r_+^2 \kappa_+} (r_+ - r)(1 + [r_+ - r, (r_+ - r) \frac{\gamma}{\gamma - 1}]_1)$$

(2.11)
as $r \to r_+ - 0$. ([TOVdS] Theorem 4)

Here and hereafter we use

**Notation 1.** The symbol $[X]_Q$ stands for various convergent power series of the form

$$\sum_{k \geq Q} a_k X^k.$$  

The symbol $[X_1, X_2]_Q, Q = 0, 1, 2, \cdots$ stands for various convergence double power series of the form

$$\sum_{k_1 + k_2 \geq Q} a_{k_1 k_2} X_1^{k_1} X_2^{k_2}.$$
2.2 Equations for the small perturbation from the equilibrium

Using the fixed equilibrium \( m = \bar{m}(r) \), we take the variable \( r \) given by its inverse function. We are going to a solution near equilibrium of the form

\[
R = r(1 + y), \quad V = rv.
\]

Here \( y, v \) are small unknowns.
The equations turn out to be

\[ e^{-F} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right) v, \quad (2.12a) \]

\[ e^{-F} \frac{\partial v}{\partial t} = \frac{(1 + y)^2}{c^2} \frac{P}{\bar{\rho}} v \frac{\partial}{\partial r} (rv) + \]

\[ -G(1 + y) \left( \frac{m}{r^3 (1 + y)^3} + \frac{4\pi}{c^2} P \right) + \frac{c^2 \Lambda}{3} (1 + y) + \]

\[ -\left(1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r (1 + y)} - \frac{\Lambda}{3} r^2 (1 + y)^2 \right) \times \]

\[ \times \left(1 + \frac{P}{c^2 \rho}\right)^{-1} (1 + y)^2 \frac{\partial P}{\bar{\rho}r} \frac{\partial}{\partial r}. \quad (2.12b) \]
Here $m = \bar{m}(r)$ is a given function and $\rho, P$ are considered as given functions of $r$ and the unknowns $y, z (:= r \partial y / \partial r)$ as follows:

\[
\rho = \bar{\rho}(r)(1 + y)^{-2}(1 + y + z)^{-1},
\]
\[
P = \bar{P}(r)(1 - \Gamma(\bar{u}(r))(3y + z) - \Phi(\bar{u}(r), y, z)).
\]

Here

\[
\Gamma := \frac{\rho}{P} \frac{dP}{d\rho}
\]

and $\Phi(u, y, z)$ is an analytic function of the form $\sum_{k_0 \geq 0, k_1 + k_2 \geq 2} u^{k_0} y^{k_1} z^{k_2}$. We shall denote such a function by $[u; y, z]_{0; 2}$ hereafter.
2.3 Analysis of the linearized equation

Let us linearize (2.12a)(2.12b):

\[ e^{-\bar{F}} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right)v, \]  \hspace{1cm} (2.13a)

\[ e^{-\bar{F}} \frac{\partial v}{\partial t} = E_2 y'' + E_1 y' + E_0 y, \]  \hspace{1cm} (2.13b)
where \( y'' = \frac{\partial^2 y}{\partial r^2}, y' = \frac{\partial y}{\partial r} \) and

\[
E_2 = e^{-2\bar{H}}(\rho + P/c^2)^{-1}P\Gamma,
\]

\[
\frac{E_1}{E_2} = \frac{d}{dr} \left( \bar{H} + \bar{F} - \log(1 + P/c^2\rho) + \log(P\Gamma r^4) \right),
\]

\[
E_0 = \frac{4\pi G}{c^2} 3(\Gamma - 1)P + 
\]

\[
+ \left( -1 - 3\Gamma e^{-2\bar{H}} + 3(\Gamma - 1)e^{-2\bar{H}}(1 + P/c^2\rho)^{-1} \right)(\rho + P/c^2)^{-1} \frac{1}{r} \frac{d\bar{P}}{dr} + 
\]

\[
+ 3e^{-2\bar{H}}(\rho + P/c^2)^{-1} \frac{1}{r} \frac{d}{dr} P\Gamma + 
\]

\[
+ \Lambda \left( c^2 + r \frac{d\bar{u}}{dr} \right).
\]
Here $\bar{X}, \overline{XXX}$ denote the evaluations along the fixed equilibrium. Putting

$$\mathcal{L}(D)y := -e^{2\bar{F}}(1 + P/c^2\rho)(E_2D^2y' + E_1Dy + E_0y),$$

we get the linearized wave equation

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L}\left(\frac{\partial}{\partial r}\right)y = 0. \quad (2.14)$$
We can rewrite $\mathcal{L}$ in the formally self-adjoint form

$$\mathcal{L}\left(\frac{d}{dr}\right)y = -\frac{1}{b} \frac{d}{d r} a \frac{d y}{d r} + Q y,$$

where

$$a = e^{\bar{H} + \bar{F}} \frac{\bar{P} \Gamma r^4}{1 + P/c^2 \rho}$$  \hspace{2cm} (2.15a)

$$b = e^{3\bar{H} - \bar{F}} \frac{r^4 \bar{\rho}}{1 + P/c^2 \rho}$$  \hspace{2cm} (2.15b)

$$Q = -e^{2\bar{F}} \frac{1}{1 + P/c^2 \rho E_0}.$$  \hspace{2cm} (2.15c)

It is easy to see that $Q$ is bounded on $0 \leq r \leq r_+$. 

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For the sake of the simplicity of discussion, suppose

**Assumption 3.** The function $\rho \mapsto P$ is analytic near $\rho_0$.

The **Assumption 3** means $\eta \mapsto \Upsilon(\eta)$ is analytic near $\eta_0 = A\rho_0^{\gamma-1}/c^2$.

We introduce the change of variables $r \leftrightarrow r$. The new variable $x$ is defined by

$$x := \frac{\tan^2 \theta}{1 + \tan^2 \theta} \quad \text{with} \quad \theta := \frac{\pi}{2\xi_+} \int_0^r \sqrt{\frac{\rho}{\Gamma P}} e^{\bar{F} + \bar{H}} dr \quad (2.16)$$

Here

$$\xi_+ := \int_0^{r^+} \sqrt{\frac{\rho}{\Gamma P}} e^{\bar{F} + \bar{H}} dr. \quad (2.17)$$

We can and shall assume that $\xi_+/\pi = 1$ without loss of generality,
by changing the scale of $ct$ or the coefficient of $e^{-\bar{F}}$.

Then $0 < r < r_+$ is mapped onto $0 < x < 1$ and there are positive constants $C_0, C_1$ such that

$$r = C_0 \sqrt{x}(1 + [x]_1) \quad \text{for} \quad 0 < x \ll 1,$$

$$r_+ - r = C_1 (1 - x)(1 + [1 - x, [1 - x]_{\gamma-1}]_1) \quad \text{for} \quad 0 < 1 - x \ll 1$$

Using this variable, we can write the operator $\mathcal{L}$ as

$$\mathcal{L} y = -x(1 - x) \frac{d^2 y}{dx^2} - \left( \frac{5}{2} (1 - x) - \frac{N}{2} x \right) \frac{dy}{dx} +$$

$$+ L_1(x) \frac{dy}{dx} + L_0(x)y,$$

(2.20)
where \( L_1(x), L_0(x) \) are smooth functions on \([0, 1]\) such that

\[
L_\mu(x) = \begin{cases} 
[x]_\mu & \text{for} \quad 0 < x \ll 1 \\
[1 - x, (1 - x)^{N/2}]_\mu & \text{for} \quad 0 < 1 - x \ll 1
\end{cases}
\]

for \( \mu = 0, 1 \). Here and hereafter we use

**Notation 2.** Put

\[
N = \frac{2\gamma}{\gamma - 1}, \quad \text{or} \quad \gamma = \frac{N}{N - 2}.
\]

We can claim
Proposition 1. The operator $T_0$, $\mathcal{D}(T_0) = C^\infty_0([0, r_+])$, $T_0 y = Ly$ in the Hilbert space $L^2([0, r_+], b(r)dr)$ admits the Friedrichs extension $\mathcal{T}$, a self adjoint operator, whose spectrum consists of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_\nu < \cdots \rightarrow +\infty$.

Hereafter we shall denote by $\mathcal{L}$ the self-adjoint operator $\mathcal{T}$ in $L^2([0, r_+], b(r)dr)$. Moreover it is easy to see, keeping in mind the behavior of $\bar{u}(r)$ as $r \rightarrow r_+$ given by (2.11), that $L^2([0, r_+], b(r)dr)$ is isomorphic to $L^2([0, 1], x^{3/2}(1 - x)^{N/2 - 1}dx)$, the space of functions of the variable $x$. In this sense we consider $\mathcal{L}$ as a self-adjoint operator acting on functions of $x$ in $L^2([0, 1], x^{3/2}(1 - x)^{N/2 - 1}dx)$ of the form (2.20).
Proposition 2. Any eigenfunction \( \psi(x) \) of \( \mathcal{L} \) is of the form

\[
\psi(x) = \begin{cases} 
C_0 + [x]_1 & \text{for } 0 < x \ll 1 \\
C_1 + [1 - x, (1 - x)^{N/2}]_1 & \text{for } 0 < 1 - x \ll 1
\end{cases}
\]

where the constants \( C_0, C_1 \) are such that \( C_0 \neq 0, C_1 \neq 0 \).

Proof is due to [OJM] *3 Lemma 2.
2.4 Rewriting the equation system (2.12a)(2.12b) using $\mathcal{L}$

Let us go back to the system of equations (2.12a)(2.12b). We are going to rewrite the system of equations (2.12a) (2.12b) as

\[
\frac{\partial y}{\partial t} - J(x, y, x \frac{\partial y}{\partial x}) v = 0, \tag{2.21a}
\]
\[
\frac{\partial v}{\partial t} + H_1(x, y, x \frac{\partial y}{\partial x}, v, \Lambda) \mathcal{L}\left( \frac{\partial}{\partial x} \right) y + H_2(x, y, x \frac{\partial y}{\partial x}, v, x \frac{\partial v}{\partial x}, \Lambda) = 0. \tag{2.21b}
\]
We shall use the analysis of $\partial P/\partial r$ given in [ssEE] *4(6.2):

$$-\frac{1}{r\bar{\rho}} \frac{\partial P}{\partial r} = -\frac{1}{r\bar{\rho}} \frac{d\bar{P}}{dr} + (1 + \partial_z \Phi/\Gamma) \frac{1}{r\bar{\rho}} \frac{\partial}{\partial r} \bar{P} \Gamma(3y + z) +$$

$$+ \frac{\bar{P}}{r\bar{\rho}} \cdot [Q0] + \frac{1}{r\bar{\rho}} \frac{d\bar{P}}{dr} \cdot [Q1],$$

where $[Q0],[Q1]$ are given by [ssEE] (6.3a), (6.3b).

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We put

the right-hand side of (2.12b) = \([R2] + [R1] + [W]\frac{\Lambda}{3}\),

where

\[ [R2] := \frac{(1 + y)^2}{c^2} \frac{P}{\bar{\rho}} v(v + w) \quad \text{with} \quad w = r \frac{\partial v}{\partial r}, \]

\[ [W] := c^2(1 + y) - r^2(1 + y)^4(1 + P/c^2 \rho)^{-1} \left( -\frac{1}{r \bar{\rho}} \frac{\partial P}{\partial r} \right). \]

We put

\[ [R1] = [R3] + [R4] + [R5] + [R6] + [R7] \]
as in [ssEE]. But the analysis of $[W]$ is new: We put

$$[W] = c^2 + r \frac{d\tilde{u}}{dr} + [W1] + [W2] + [W3] + [W4],$$

$$[W1] := c^2 y - r^2 (1 + y)^2 (1 + \frac{P}{c^2 \rho})^{-1} \left( - \frac{1}{r \tilde{\rho}} \frac{d\tilde{P}}{dr} \right) - r \frac{d\tilde{u}}{dr}$$

$$= [W1L] + [W1Q],$$

$$[W1L] := c^2 y - 4r \frac{d\tilde{u}}{dr} y + r \left( \frac{P}{\rho + P/c^2} \right) (\Gamma - 1)(3y + z) \frac{d\tilde{u}}{dr},$$

$$[W2] := -r^2 (1 + y)^4 (1 + \frac{P}{c^2 \rho})^{-1} (1 + \partial_z \Phi / \Gamma) \frac{1}{r \tilde{\rho}} \frac{\partial}{\partial r} \tilde{P} (3y + z),$$

$$[W3] := -r^2 (1 + y)^4 (1 + \frac{P}{c^2 \rho})^{-1} \frac{\tilde{P}}{r \tilde{\rho}} [Q0],$$

$$[W4] := -r^2 (1 + y)^4 (1 + P/c^2 \rho)^{-1} \frac{1}{r \tilde{\rho}} \frac{d\tilde{P}}{dr} [Q1].$$
Then it follows from (2.4b) that

\[
[R1] + [W] \frac{\Lambda}{3} = [R3L] + [R3Q] + [R4L] + [R4Q] + [R5] + [R6] + [R7] + \\
+ ([W1L] + [W1Q] + [W2] + [W3] + [W4]) \frac{\Lambda}{3}.
\]

Let us define

\[
1 + G_1 = (1 + \partial_z \Phi / \Gamma) \left( 1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r(1 + y)} - r^2 (1 + y)^2 \frac{\Lambda}{3} \right) \times \\
\times \left( 1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^3 \right)^{-1} \frac{1 + \frac{P/c^2 \rho}{1 + P/c^2 \rho}}{1 + y^2}.
\]
Then we have

\[-e^{-2F}(1 + P/c^2\rho)^{-1}Ly = [R3L] + [R4L] + [W1L]\frac{\Lambda}{3} +
\]
\[+ \frac{1}{1 + G_1} ([R5] + [W2] \frac{\Lambda}{3}).\]

Putting

\[G_2 := G_1 ([R3L] + [R4L] + [W1L] \frac{\Lambda}{3}) +
\]
\[- ([R3Q] + [R4Q] + [R6] + [R7] + [R2]) +
\]
\[- ([W1Q] + [W3] + [W4]) \frac{\Lambda}{3},\]

\[H_2 := e^F G_2,\]
\[H_1 := e^{F-2F}(1 + P/c^2\rho)^{-1}(1 + G_1),\]
we can write

\[ e^F \times \text{(the right-hand side of (2.12b))} = -H_1 \mathcal{L}y - H_2. \]

Putting

\[ J := e^F (1 + P/c^2 \rho), \]

we rewrite the system of equations (2.12a)(2.12b) as (2.21a)(2.21b).
We see that the functions

\[
J : (x, y, z) \mapsto J(x, y, z, \Lambda) \\
H_1 : (x, y, z, v) \mapsto H_1(x, y, z, v, \Lambda) \\
H_2 : (x, y, z, v, w) \mapsto H_2(x, y, z, v, w, \Lambda),
\]

\(\Lambda\) being fixed, enjoy the following properties (B.1), (B.2), (B.3):
(B.1): \( J, H_1, H_2 \) belong to the functional classes \( \mathfrak{A}^0_{(N)}(U^2), \mathfrak{A}^0_{(N)}(U^3), \mathfrak{A}^2_{(N)}(U^4) \), respectively.

Here we use

**Definition 2.** By \( \mathfrak{A}^Q_{(N)}(U^p) \), \( U \) being the interval \( ] - \delta, \delta [ , 0 < \delta \ll 1 \), we denote the set of all smooth functions \( f \) defined on \( [0,1] \times U^p \) such that there are convergent power series

\[
\Phi_0(X, Y_1, \ldots, Y_p) = \sum_{|\vec{k}| \geq Q} a_{j\vec{k}} X^j Y_1^{k_1} \cdots Y_p^{k_p}
\]

and

\[
\Phi_1(X_1, X_2, Y_1, \ldots, Y_p) = \sum_{|\vec{k}| \geq Q} b_{j_1j_2\vec{k}} X_1^{j_1} X_2^{j_2} Y_1^{k_1} \cdots Y_p^{k_p}
\]
such that

\[
f(x, y_1, \ldots, y_p) = \Phi_0(x, y_1, \ldots, y_p) \quad \text{for} \quad 0 < x < 1
\]

and

\[
f(x, y_1, \ldots, y_p) = \Phi_1(1-x, (1-x)^{N/2}, y_1, \ldots, y_p) \quad \text{for} \quad 0 < 1-x < 1.
\]
(B.2): It holds that

$$J(x, 0, 0)H_1(x, 0, 0, 0) = 1,$$

(2.22)

and

$$\frac{1}{C} < J(x, 0, 0) < C$$

(2.23)

with a sufficiently large $C$. 
(B.3): It holds that

\[ \frac{\partial J}{\partial z} \equiv_{(N)} 0, \quad (2.24a) \]
\[ \left( \frac{\partial H_1}{\partial z} \right) \mathcal{L} y + \frac{\partial H_2}{\partial z} \equiv_{(N)} 0, \quad (2.24b) \]
\[ \frac{\partial H_2}{\partial w} \equiv_{(N)} 0. \quad (2.24c) \]

Here we mean

**Notation 3.** For \( f \in \mathfrak{A}^0_{(N)}(U^p) \), \( f \equiv_{(N)} 0 \) means that there exists a convergent power series \( \Phi(X_1, X_2, Y_1, \cdots, Y_p) \) such that

\[ f(x, y_1, \cdots, y_p) = (1-x)\Phi(1-x, (1-x)^{N/2}, y_1, \cdots, y_p) \quad \text{for} \quad 0 < 1-x \ll 1. \]
Proof of (2.24b):

\[(\partial_z H_1)\mathcal{L}y + \partial_z H_2 \equiv_{(N)} 0\]

is similar to that of [ssEE] Proposition 11.

Actually we see

\[(\partial_z H_1)\mathcal{L}y + \partial_z H_2 \equiv_{(N)} e^F[S]\]

and we have to show \([S] \equiv_{(N)} 0\), where

\[S := (\partial_z G_1)\left(e^{-2\bar{F}}(1 + P/c^2\rho)^{-1}\mathcal{L}y + [R3L] + [R4L] + [W1L]\Lambda/3\right) + \]
\[+ G_1 \partial_z([R3L] + [R4L] + [W1L]\Lambda/3) + \]
\[- \partial_z\left([R3Q] + [R4Q] + [R6] + [R7] + [R2] + ([W1Q] + [W3] + [W4])\Lambda/3\right).\]
But we have

\[ [S] \equiv_{(N)} - \frac{\partial_z G_1}{1 + G_1} \left( [R5] + [W2] \frac{\Lambda}{3} \right) - \partial_z \left( [R7] + [W4] \frac{\Lambda}{3} \right), \]

since \( \partial_z[R3L], \partial_z[R4L], \partial_z[R3Q], \partial_z[R4Q], \partial_z[R6], \partial_z[R2], \partial_z[W1L],\)
\( \partial_z[W1Q], \partial_z[W3] \) are all \( \equiv_{(N)} 0 \) clearly. By a tedious calculation, we get

\[ -\frac{\partial_z G_1}{1 + G_1} \left( [R5] + [W2] \frac{\Lambda}{3} \right) \equiv \partial_z \left( [R7] + [W4] \frac{\Lambda}{3} \right) \]
\[ \equiv_{(N)} -\partial_z^2 \Phi \left( 1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r (1 + y)} - r^2 (1 + y)^2 \frac{\Lambda}{3} \right) (1 + y)^2 \frac{1}{r \bar{\rho}} \frac{d \bar{P}}{dr} (3y + z), \]

so that \([S] \equiv_{(N)} 0\). This completes the proof.
2.5 Main results

Let us fix a time periodic solution of the linearized equation:

\[ Y_1 = \sin(\sqrt{\lambda} t + \Theta_0) \psi(x), \]

where \( \lambda \) is a positive eigenvalue of the operator \( \mathcal{L} \), \( \psi \) is an associated eigenfunction and \( \Theta_0 \) is an arbitrary constant. We seek a solution of the form

\[
\begin{bmatrix}
  y \\
  v
\end{bmatrix}
= \begin{bmatrix}
  \varepsilon(Y_1 + \ddot{y}) \\
  \varepsilon(V_1 + \ddot{v})
\end{bmatrix}
= \varepsilon \begin{bmatrix}
  Y_1 \\
  V_1
\end{bmatrix} + \varepsilon \vec{w},
\]

(2.25)

where

\[ V_1 = e^{-\bar{F}} (1 + \frac{P}{c^2 \rho})^{-1} \frac{\partial Y_1}{\partial t}. \]
Then the equation to be solved turns out to be

$$\mathcal{P}(\vec{w}) = \varepsilon \vec{c}, \quad (2.26)$$

for the unknown $\vec{w} = (\hat{y}, \hat{v})^\top$. Here

$$\mathcal{P}(\vec{w}) - \varepsilon \vec{c} = \frac{1}{\varepsilon} \begin{bmatrix}
\text{the left-hand side of (2.21a)} \\
\text{the left-hand side of (2.21b)}
\end{bmatrix}_{y = \varepsilon(Y_1 + \hat{y}), v = \varepsilon(V_1 + \hat{v})}.$$

We are seeking the solution $\vec{w}$ satisfying the initial condition

$$\vec{w}|_{t=0} = \begin{bmatrix}
\hat{y} \\
\hat{v}
\end{bmatrix}_{t=0} = \vec{0}. \quad (2.27)$$
The Fréchet derivative $D\Psi(\vec{w})$ of the nonlinear operator $\Psi$ at $\vec{w}$:

$$D\Psi(\vec{w})\vec{h} = \begin{bmatrix} DP1 \\ DP2 \end{bmatrix}, \quad \text{with} \quad \vec{h} = \begin{bmatrix} h \\ k \end{bmatrix}$$

is given by

$$DP1 = \frac{\partial h}{\partial t} - Jk - \left( (\partial_y J)v + (\partial_z J)vx \frac{\partial}{\partial x} \right)h$$

$$DP2 = \frac{\partial k}{\partial t} + H_1\mathcal{L}h +
+ \left( (\partial_y H_1)\mathcal{L}y + \partial_y H_2 + ((\partial_z H_1)\mathcal{L}y + \partial_z H_2)x \frac{\partial}{\partial x} \right)h +
+ \left( (\partial_v H_1)\mathcal{L}y + \partial_v H_2 + \partial_w H_2x \frac{\partial}{\partial x} \right)k.$$

Thanks to (B.3) we can claim that there are functions $a_{01}$, $a_{00}$, $a_{11}$,
$a_{10}, a_{21}, a_{20}$ of $t, x, y, \partial_x y, \partial_x^2 y, v, \partial_x v$ in the class $\mathfrak{A}_N([0, T] \times U^5)$ such that

\[
DP1 = \frac{\partial h}{\partial t} - Jk + \left( a_{01} x (1 - x) \frac{\partial}{\partial x} + a_{00} \right) h,
\]

\[
DP2 = \frac{\partial k}{\partial t} + H_1 \mathcal{L} h + \left( a_{11} x (1 - x) \frac{\partial}{\partial x} + a_{10} \right) h + \left( a_{21} x (1 - x) \frac{\partial}{\partial x} + a_{20} \right) k.
\]

Here we use

**Definition 3.** By $\mathfrak{A}_N([0, T] \times U^p)$ we denote the set of all smooth functions $a(t, x, y_1, \cdots, y_p)$ of $t \in [0, T], x \in [0, 1[, y_j \in U = ]-\delta, \delta[, 0 < \delta \ll 1$, such that there are analytic functions $\Psi_0$ on $[0, T] \times U^{p+1}$, $\Psi_1$ on $[0, T] \times U^{p+2}$ such that

\[
a(t, x, y_1, \cdots) = \begin{cases} 
\Psi_0(t, x, y_1, \cdots) & \text{for } 0 < x \ll 1 \\
\Psi_1(t, 1 - x, (1 - x)^{N/2}, y_1, \cdots) & \text{for } 0 < 1 - x \ll 1
\end{cases}
\]
In fact (B.3) plays a crucial rôle. Otherwise the factor \((1 - x)\) in terms \(a_{01}x(1 - x)\frac{\partial}{\partial x}, a_{11}x(1 - x)\frac{\partial}{\partial x}, a_{21}x(1 - x)\frac{\partial}{\partial x}\) would lack so that these terms lacking the factor \(1 - x\) would invade the principal part of the linearized operator \(\mathcal{L}\).
We suppose that either one of the following assumptions holds:

**Assumption 4.** \( \frac{1}{\gamma - 1} \) is an integer.

**Assumption 5.** It holds that \( 1 < \gamma < \frac{54}{53} \).

Then we can apply the Nash-Moser theorem to get the main results, since the Fréchet derivative \( D\mathcal{P}(\bar{w}) \) has the inverse, say, the resolution of the wave equation, on a suitable graded functional spaces of functions \( \mathbf{h} \) such that \( \mathbf{h}|_{t=0} = 0 \). We employ the Nash-Moser theorem, since this resolution involves so called regularity loss at the vacuum boundary \( x = 1 \), while the singularity at the center \( x = 0 \) does not cause regularity
loss because of the factor $x$ in terms $z = x \frac{\partial y}{\partial x}, w = x \frac{\partial v}{\partial x}$.

Namely, under the Assumption 4, when $N/2$ is an integer, we apply the Nash-Moser theorem formulated by R. S. Hamilton given in [H] *5 as [ssEE], while, under the Assumption 5, when $N > 108$, we apply the Nash-Moser theorem formulated by J. T. Schwartz in [Sch] *6 as [JDE2017] *7.

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Why Nash-Moser? – [ Regularity loss $\leadsto$ Iteration invalid ]

• $2\sqrt{X} = \xi = |\xi|, \xi \in \mathbb{R}^N \Rightarrow$

\[
X \frac{d^2}{dX^2} + \frac{N}{2} \frac{d}{dX} = \frac{d^2}{d\xi^2} + \frac{N-1}{\xi} \frac{d}{d\xi} = \triangle, \quad \frac{d}{dX} = \frac{2}{\xi} \frac{d}{d\xi} \sim 2 \frac{d^2}{d\xi^2}
\]

•

\[
\frac{\partial^2 U}{\partial t^2} - \triangle U = f(t, \xi), \quad U|_{t=0} = \frac{\partial U}{\partial t} \bigg|_{t=0} = 0,
\]

\[
f \in C([0, T], H^p(\mathbb{R}^N)) \quad \Rightarrow \quad U \in C([0, T], H^{p+1}(\mathbb{R}^N))
\]
The results are following:

**Theorem 1.** Given $T > 0$ under Assumption 4, there is a positive number $\epsilon_0$ such that, for $|\epsilon| \leq \epsilon_0$, there is a solution $\vec{w} \in C^\infty([0,T] \times [0,1])$ of (2.26)(2.27) such that

$$\sup_{j+k \leq n} \left\| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^k \vec{w} \right\|_{L^\infty([0,T] \times [0,1])} \leq C(n)|\epsilon|, \quad \forall n \in \mathbb{N},$$

and hence a solution $(y, v)$ of (2.12a)(2.12b) of the form $y = \epsilon Y_1 + O(\epsilon^2), \ v = \epsilon V_1 + O(\epsilon^2)$. 
Theorem 2. Given $T > 0$ under Assumption 5, there is a positive number $\epsilon_0$ such that, for $|\epsilon| \leq \epsilon_0$, there is a solution $\vec{w}$ of (2.26)(2.27) such that

$$\|\vec{w}\|_{s_N+1} \leq C\epsilon,$$

where $s_N = \left[ \frac{N}{2} \right] + 1 = \min \{ s \in \mathbb{N} \mid s > \frac{N}{2} \}$.

Here

**Definition 4.** We put

$$\|\vec{w}\|_{\nu}^2 = \|y\|_{\nu}^2 + \|v\|_{\nu}^2 \quad \text{for} \quad \vec{w} = (y, v)^\top,$$
where

\[ \|u\|_{\nu}^2 = \sum_{i+\kappa \leq \nu} \int_0^T (\|(-\partial_t)^{2i} u(t, \cdot)\|_{\kappa}^*)^2 \, dt, \]

\[ (\|u\|_{\kappa}^*)^2 = (\|u[0]\|_{[0]\kappa}^*)^2 + (\|u[1]\|_{[1]\kappa}^*)^2, \]

\[ (\|u[\mu]\|_{[\mu]\kappa}^*)^2 = \sum_{0 \leq m \leq \kappa} \|\triangle_m [\mu] u[\mu]\|_{[\mu]}^2, \quad \mu = 0, 1 \]

\[ \triangle[0] = x \frac{d^2}{dx^2} + \frac{5}{2} \frac{d}{dx}, \quad \triangle[1] = X \frac{d^2}{dX^2} + \frac{N}{2} \frac{d}{dX} \quad \text{for} \quad X = 1 - x, \]

\[ \|f\|_{[0]}^2 = \int_0^1 |f(x)|^2 x^{3/2} \, dx, \]

\[ \|f\|_{[1]}^2 = \int_0^1 |f(x)|^2 (1 - x)^{N/2 - 1} \, dx. \]
Here

\[ u^{[0]}(x) = \omega(x)u(x), \quad u^{[1]}(x) = (1 - \omega(x))u(x) \]

with a cut-off function \( \omega \in C^\infty(\mathbb{R}, [0, 1]) \) such that \( \omega(x) = 1 \) for \( 0 < x \ll 1 \) and \( \omega(x) = 0 \) for \( 0 < 1 - x \ll 1 \).
Note that

\[ R(t, r_+) = r_+ (1 + \varepsilon \sin(\sqrt{\lambda} t + \Theta_0) + O(\varepsilon^2)) , \]

provided that \( \psi \) has been normalized as \( \psi(x = 1) = 1 \), and that the density distribution enjoys the ‘physical vacuum boundary’ condition:

\[
\rho(t, r) = \begin{cases} 
C(t)(r_+ - r)^{\frac{1}{\gamma - 1}} (1 + O(r_+ - r)) & (0 \leq r < r_+) \\
0 & (r_+ \leq r) 
\end{cases}
\]

with a smooth function \( C(t) \) of \( t \) such that

\[
C(t) = \left( \frac{\gamma - 1}{A \gamma} \frac{Q_+}{r_+^2 \kappa_+} \right)^{\frac{1}{\gamma - 1}} + O(\varepsilon) .
\]
Also we can consider the Cauchy problem

\[
\begin{align*}
\frac{\partial y}{\partial t} - Jv &= 0, \\
\frac{\partial v}{\partial t} + H_1 \mathcal{L}y + H_2 &= 0, \\
y \big|_{t=0} &= \psi_0(x), \\
v \big|_{t=0} &= \psi_1(x).
\end{align*}
\]

Then we have

**Theorem 3.** *Given* \( T > 0 \), *there exits a small positive* \( \delta \) *such that if* \( \psi_0, \psi_1 \in C^\infty([0, 1]) \) *satisfy

\[
\max_{k \leq \mathcal{K}} \left\{ \left\| \left( \frac{d}{dx} \right)^k \psi_0 \right\|_{L^\infty}, \left\| \left( \frac{d}{dx} \right)^k \psi_1 \right\|_{L^\infty} \right\} \leq \delta,
\]

*then there exists a unique solution* \((y, v)\) *of the Cauchy problem in* \( C^1([0, T] \times [0, 1]) \). *Here* \( \mathcal{K} \) *is sufficiently large number.*
2.6 Metric in the exterior domain

Let us consider the moving solutions constructed in the preceding discussion, which are defined on \( 0 \leq t \leq T, 0 < r \leq r_+ \). We discuss on the extension of the metric onto the exterior vacuum region \( r > r_+ \).
Keeping in mind the Birkhoff’s theorem, we try to patch the Schwarzschild-de Sitter metric

\[ ds^2 = \kappa^\# c^2 (dt^\#)^2 - \frac{1}{\kappa^\#} (dR^\#)^2 - (R^\#)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

from the exterior region. Here \( t^\# = t^\#(t, r), R^\# = R^\#(t, r) \) are smooth functions of \( 0 \leq t \leq T, r_+ \leq r \leq r_+ + \delta \), \( \delta \) being a small positive number, and

\[ \kappa^\# = 1 - \frac{2Gm_+}{c^2 R^\#} - \frac{\Lambda}{3} (R^\#)^2. \]

The patched metric is

\[ ds^2 = g_{00} c^2 dt^2 + 2g_{01} c dt dr + g_{11} dr^2 + g_{22} (d\theta^2 + \sin^2 \theta d\phi^2), \]
where

\[ g_{00} = \begin{cases} 
  e^{2F} = \kappa_+ e^{-2u/c^2} & (r \leq r_+) \\
  \kappa^\#(\partial_t t^\#)^2 - \frac{1}{c^2 \kappa^\#}(\partial_t R^\#)^2 & (r_+ < r) 
\end{cases} \]

\[ g_{01} = \begin{cases} 
  0 & (r \leq r_+) \\
  c\kappa^\#(\partial_t t^\#)(\partial_r t^\#) - \frac{1}{c\kappa^\#}(\partial_t R^\#)(\partial_r R^\#) & (r_+ < r) 
\end{cases} \]

\[ g_{11} = \begin{cases} 
  -e^{2H} = -\left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-1} (\partial_r R)^2 & (r \leq r_+) \\
  c^2 \kappa^\#(\partial_r t^\#)^2 - \frac{1}{\kappa^\#}(\partial_r R^\#)^2 & (r_+ < r) 
\end{cases} \]

\[ g_{22} = \begin{cases} 
  -R^2 & (r \leq r_+) \\
  -(R^\#)^2 & (r_+ < r). 
\end{cases} \]

We require that \( R = R^\# \) and \( \partial_r R = \partial_r R^\# \) along \( r = r_+ \). It is nec-
necessary for that $g_{22}$ is of class $C^1$. Moreover, as [ssEE] Supplementary Remark 4, we see that

$$\frac{\partial t^#}{\partial t}, \frac{\partial t^#}{\partial r}, \frac{\partial^2 t^#}{\partial r^2}, \frac{\partial^2 R^#}{\partial r^2}$$

at $r = r_+ + 0$ are uniquely determined so that $g_{\mu\nu}$ are of class $C^1$ across $r = r_+$. 
By a tedious calculation we have

\[ \frac{\partial^2 R^\#}{\partial r^2} \bigg|_{r_+0} - \frac{\partial^2 R}{\partial r^2} \bigg|_{r_-0} = A \left( \frac{\partial R}{\partial r} \right)^2, \]

where

\[ A = - \frac{V^2}{c^2} \left[ \left( \frac{Gm_+}{c^2 R^2} - \frac{\Lambda}{3} R + \frac{1}{\sqrt{\kappa_+}} \frac{1}{c^2} \frac{\partial V}{\partial t} \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-2} \right]_{r=r+0}. \]

Since \( \partial R/\partial r \) is near to 1 and

\[ \left[ \left( \frac{Gm_+}{c^2 R^2} - \frac{\Lambda}{3} R + \frac{1}{\sqrt{\kappa_+}} \frac{1}{c^2} \frac{\partial V}{\partial t} \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-2} \right]_{r=r+0} \]

is near to \( \frac{Q_+}{c^2 r_+^2 \kappa_+^2} \neq 0 \), we see that
\[ \frac{\partial^2 R^\#}{\partial r^2} \equiv \frac{\partial^2 R}{\partial r^2}, \text{ that is, the patching is twice continuously differentiable, if and only if } V \equiv 0 \text{ at } r = r_+, \text{ which is the case if and only if the solution under consideration is a static equilibrium.} \]
3 Axially symmetric problem

In this section we consider the Einstein-Euler equations with $\Lambda = 0$.

3.1 Slowly rotating axisymmetric solutions in the non-relativistic problem

In the non-relativistic Newtonian theory, the interior structures of gaseous stars are governed by the Euler-Poisson equations. In recent
works [JJTM1] *8 and [JJTM.2] *9 we have constructed uniformly and slowly rotating axisymmetric solutions to the Euler-Poisson equations.


We try to extend the results of the non-relativistic problem to the relativistic problem governed by the Einstein-Euler equations. Actually, for the Einstein-Euler equations, it seems that no existence theorem of the axisymmetric stationary metric in the interior of gaseous stars in the presence of the pressure, say, neither of vacuum nor of dust, has been established, except only one found in the work by U. Heilig [Heilig]*10, although many interesting results have been obtained for the vacuum region, e.g., the Kerr’s metric and so on. See e.g. [Kerr]*11, [KerrGeo]*12

---

Result for the non-relativistic problem

Assuming \( P = A \rho^\gamma \) with \( 6/5 < \gamma < 2 \), we constructed a solution with density distribution

\[
\rho = \rho_0 \left( \Theta \left( \frac{r}{a}, \zeta; \frac{1}{\gamma - 1}, b \right) \lor 0 \right)^{\frac{1}{\gamma-1}},
\]

where \( \Theta \lor 0 = \max(\Theta, 0) \), with the velocity field

\[
\vec{v} = (-\Omega y, \Omega x, 0)^T
\]

of the Euler-Poisson equations

\[
\frac{\partial \rho}{\partial t} + (\nabla | \rho \vec{v} |) = 0,
\]

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla) \vec{v} \right) + \nabla P = -\rho \nabla \Phi,
\]

\[
\Delta \Phi = 4\pi G \rho,
\]
where \((x, y, z) \in \mathbb{R}^3\), \(r = \sqrt{x^2 + y^2 + z^2}\), \(\zeta = z/r\), and \(\Omega\) is a constant, the angular velocity of the uniform rotation. The last Poisson equation is replaced by the Newton potential

\[
\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}'
\]

provided that the density \(\rho\) is compactly supported.

The positive number \(\rho_0\) is the central density, an arbitrary positive constant, and the parameters \(a, b\) are defined as

\[
a = \sqrt{\frac{A\gamma}{4\pi G(\gamma - 1)} \rho_0^{\frac{2-\gamma}{2}}} , \quad b = \frac{\Omega^2}{2\pi G \rho_0}.
\]

We require that \(b\) is sufficiently small, say, \(b \leq \varepsilon_0\).
Here the function $\Theta(r, \zeta; \frac{1}{\gamma-1}, b)$ is the ‘**distorted Lane-Emden function**’ with the following properties:

1) The function $\Theta^b : (x, y, z) \mapsto \Theta(r, \zeta; \frac{1}{\gamma-1}, b)$ is an equatorially and axially symmetric $C^2$-function on the domain $\bar{B}(\Xi_0) := \{(x, y, z) \in \mathbb{R}^3 | r \leq \Xi_0 \}$; Here $\Xi_0$ is a large positive number, which will be fixed in this article, such that $\Xi_0 \geq 2 \xi_1(\frac{1}{\gamma-1})$, $\xi_1(\frac{1}{\gamma-1})$ being the zero of the Lane-Emden function $\theta(r; \frac{1}{\gamma-1})$ of index $\frac{1}{\gamma-1}$, that is, the solution of

$$-\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\theta}{dr} = (\theta \vee 0) \frac{1}{\gamma-1}, \quad \theta|_{r=0} = 1;$$

2) $\Theta(0, \zeta; \frac{1}{\gamma-1}, b) = 1$ and there is a curve $\zeta \in [-1, 1] \mapsto r = \Xi_1(\zeta; \frac{1}{\gamma-1}, b)$ such that $\Xi_1(\zeta; \frac{1}{\gamma-1}, b) < 2 \xi_1(\frac{1}{\gamma-1})$ and

$$0 \leq r \leq \Xi_0, 0 < \Theta(r, \zeta; \frac{1}{\gamma-1}, b) \quad \Leftrightarrow \quad 0 \leq r < \Xi_1(\zeta; \frac{1}{\gamma-1}, b).$$
We note that \( u = \Theta^b \) is the equatorially and axially symmetric solution of the integral equation

\[
    u = \frac{b}{4} (x^2 + y^2) + \mathcal{G}(u),
\]

(3.1)

where

\[
    \mathcal{G}(u) = \mathcal{K}^{(3)}(u \lor 0) \frac{1}{\gamma - 1} - \mathcal{K}^{(3)}(u \lor 0) \frac{1}{\gamma - 1} (O) + 1,
\]

\[
    \mathcal{K}^{(3)} g(\vec{x}) = \frac{1}{4\pi} \int \frac{g(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}'.
\]

See Theorem 1 of [JJTM1] and Theorem 2 of [JJTM2]. The existence of \( \Theta \) is established by the fact that the Fréchet derivative

\[
    D\mathcal{G}(u) h = \mathcal{K}^{(3)} \left[ \frac{1}{\gamma - 1} (u \lor 0) \frac{1}{\gamma - 1}^{-1} h \right] - \mathcal{K}^{(3)} \left[ \frac{1}{\gamma - 1} (u \lor 0) \frac{1}{\gamma - 1}^{-1} h \right] (O)
\]

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of the operator $\mathcal{G}$ in the Banach space of equatorially and axially symmetric continuous functions on $\bar{B}(\Xi_0)$ enjoys the condition

$$\text{Ker}(I - D\mathcal{G}(u)) = \{0\}$$  \hspace{1cm} (3.2)

at $u = \theta(r; \frac{1}{\gamma-1})$, the non-rotating spherically symmetric Lane-Emden function. Based on this fact the implicit function theorem guarantees, near $\theta$, the existence of the solution of (3.1), $\Theta(r, \zeta; \frac{1}{\gamma-1}, b)$, for which the condition (3.2) holds, too. This property will play an important rôle later.
Let us fix such a solution and denote $\rho = \rho_N$ with $\rho_O = \rho_{NO}$ and put

$$u_N = u_O \Theta \left( \frac{r}{a}, \zeta; \frac{1}{\gamma - 1}, b \right) \quad \text{with} \quad u_O = \frac{A\gamma}{\gamma - 1} \rho_{NO}^{\gamma - 1}. \quad (3.3)$$

Let us denote by $\Phi_N$ the gravitational potential of the density distribution $\rho_N$, that is,

$$\Phi_N(\varpi, z) = -G \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{2\pi} \frac{d\phi'}{\sqrt{\varpi'^2 + \varpi^2 - 2\varpi\varpi' \cos \phi' + (z - z')^2}} \times \rho_N(\varpi', z') \varpi' d\varpi' dz',$$

where

$$\varpi = \sqrt{x^2 + y^2}, \quad x = \varpi \cos \phi, \quad y = \varpi \sin \phi.$$
We are looking for axisymmetric solutions of the relativistic Einstein-Euler equations which approach to this solution of the Euler-Poisson equations as $c \to +\infty$. 
3.2 Main result

Looking for axisymmetric solutions of the relativistic Einstein-Euler equations which approach to the fixed solution of the Euler-Poisson equations as $c \to +\infty$, we can describe the main conclusion of this study as follows:

We are looking for a metric of the form

$$ds^2 = e^{2F'}(cdt + A'd\phi')^2 - e^{-2F'}[e^{2K'}(d\varpi^2 + dz^2) + \Pi^2(d\phi')^2],$$

(3.4)

where $F', A', K', \Pi$ and the density distribution $\rho$ depend only on $\varpi$ and $z$. Here the co-ordinates are $x^0 = ct, x^1 = \varpi, x^2 = \phi, x^3 = z$, and $\phi' = \phi - \Omega t$, while $\Omega$ is the constant, which de-
terminated the solution of the non-relativistic equations above fixed. The 4-velocity field looked for is

\[ U^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{c e^{F'}} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi'} \right). \]

Let us fix an arbitrarily large positive number \( R \) with \( 2a\xi_1 < R \leq a\Xi_0 \). If \( u_\odot/c^2 \) is sufficiently small, then we can construct the metric and the density distribution on the domain \( \mathcal{D} = \{(\varpi, z) \mid r = \sqrt{\varpi^2 + z^2} < R \} \) such that \( F', A', K', \Pi, u \in C^{2,\alpha}(\mathcal{D}) \) and

\[ F' = -\frac{\Omega^2}{2c^2} \varpi^2 + \frac{\Phi_N}{c^2} + O\left(\frac{u_\odot^2}{c^4}\right), \quad K' = -\frac{\Omega^2}{2c^2} \varpi^2 + O\left(\frac{u_\odot^2}{c^4}\right), \]

\[ A' = -\frac{\Omega}{c} \varpi^2 \left(1 + O\left(\frac{u_\odot}{c^2}\right)\right), \quad \Pi = \varpi \left(1 + O\left(\frac{u_\odot^2}{c^4}\right)\right), \]
and

\[ \rho = \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma-1}} (u \vee 0)^{\frac{1}{\gamma-1}} \left( 1 + \Upsilon_{\rho}(u/c^2) \right) \quad \text{with} \quad u = u_N + O\left( \frac{u_0^2}{c^2} \right). \]

Of course we take \( u \) so that \( u(0,0) = u_N(0,0) = u_0 \). Actually the interior of the star \( \{ \rho > 0 \} \) is of the form \( \{ (\varpi, z) | r < r_+(\zeta) \} \) determined by a suitable curve \( r = r_+(\zeta) = a(\Xi_1(\zeta) + O(u_0^2/c^2)) \).
( Of course \( r = \sqrt{\varpi^2 + z^2}, \zeta = z/r \).)
3.3 Basic equations

Taking the coordinates

\[ x^0 = ct, \quad x^1 = \varpi, \quad x^2 = \phi, \quad x^3 = z. \quad (3.5) \]

Let us write the metric \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) in the following form (Lewis 1932 [Lewis]*13 Papapetrou 1966 [Papapetrou]*14):

\[
ds^2 = e^{2F} (c dt + A d\phi)^2 - e^{-2(F-K)} (d\varpi^2 + dz^2) - e^{-2F} \Pi^2 d\phi^2, \quad (3.6)\]

where the functions \( F, A, K \) and \( \Pi \) depend only on \( \varpi \) and \( z \).

---

The expression (3.6) is called ‘Lanczos form’, and the 4-velocity is:

\[ U^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{ce^G} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right), \]

where

\[ e^{2F} \left( 1 + \frac{\Omega}{c} A \right)^2 - e^{-2F} \frac{\Omega^2}{c^2} \Pi^2 = e^{2G} \]

for \( U_\mu U^\mu = 1 \)
We shall use the ‘corotating co-ordinate system’ characterized by

\[ t' = t, \quad \varpi' = \varpi, \quad \phi' = \phi - \Omega t, \quad z' = z. \]  

(3.7)

Since we see \( G = F' \), if we try to find \( F', K', A', \Pi \) as functions of \( \varpi, z \), then it should be satisfied

\[ (C1): \quad e^{2F'} \left( 1 - \frac{\Omega}{c} A' \right)^2 - e^{-2F'} \frac{\Omega^2}{c^2} \Pi^2 > 0. \]

Only if so, we can define \( F, K, A \) by

\[ e^{2F} = e^{2F'} \left( 1 - \frac{\Omega}{c} A' \right)^2 - e^{-2F'} \frac{\Omega^2}{c^2} \Pi^2, \quad (3.8) \]

\[ e^{2F} A = e^{2F'} \left( 1 - \frac{\Omega}{c} A' \right) A' - e^{-2F'} \frac{\Omega^2}{c^2} \Pi^2, \quad (3.9) \]
to recover (3.6).
Using the Euler equations $\nabla_\mu T^{\mu 1} = \nabla_\mu T^{\mu 3} = 0$, we have that, defining the ‘relativistic enthalpy density’ $u$ by

$$u := c^2 \int_0^\rho \frac{dP}{c^2 \rho + P} = \int_0^\rho \frac{dP}{\rho + P/c^2},$$

we have

$$F' = G = -\frac{u}{c^2} + \text{Const.} \quad (3.11)$$

Note that the other Euler equations

$$\nabla_\mu T^{\mu 0} = 0, \quad \nabla_\mu T^{\mu 2} = 0$$

hold automatically.
The Einstein’s equations are

\[ R_{\mu\nu} = \frac{8\pi G}{c^4} \mathcal{F}_{\mu\nu}, \quad \text{where} \quad \mathcal{F}_{\mu\nu} := T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T. \]  

(3.12)
The components $\mathfrak{T}_{\mu\nu}$ turn out to be as following (other $\mathfrak{T}_{\mu\nu}$’s are zero):

\begin{align*}
\mathfrak{T}_{00} &= \frac{1}{2} (c^2 \rho + P) e^{-2G} \left[ \left( f - \frac{\Omega}{c} k \right)^2 + \frac{\Omega^2}{c^2} \Pi^2 \right] + Pf, \tag{3.13a} \\
\mathfrak{T}_{02} = \mathfrak{T}_{20} &= \frac{1}{2} (c^2 \rho + P) e^{-2G} \left[ -k f - 2 \frac{\Omega}{c} fl + \frac{\Omega^2}{c^2} kl \right] -Pk, \tag{3.13b} \\
\mathfrak{T}_{22} &= \frac{1}{2} (c^2 \rho + P) e^{-2G} \left[ \Pi^2 + \left( k + \frac{\Omega}{c} l \right)^2 \right] -Pl, \tag{3.13c} \\
\mathfrak{T}_{11} = \mathfrak{T}_{33} &= \frac{e_m}{2} (c^2 \rho - P). \tag{3.13d}
\end{align*}

Note that

\[ T = c^2 \rho - 3P. \tag{3.14} \]
Thus the Einstein’s equations are, except for $0 = 0$,

\[ R_{00} = \frac{8\pi G}{c^4} \mathcal{T}_{00}, \tag{3.15a} \]
\[ R_{02} = \frac{8\pi G}{c^4} \mathcal{T}_{02}, \tag{3.15b} \]
\[ R_{22} = \frac{8\pi G}{c^4} \mathcal{T}_{22}, \tag{3.15c} \]
\[ R_{11} = \frac{8\pi G}{c^4} \mathcal{T}_{11}, \tag{3.15d} \]
\[ R_{33} = \frac{8\pi G}{c^4} \mathcal{T}_{33}, \tag{3.15e} \]
\[ R_{13} = R_{31} = 0 \tag{3.15f} \]
Proposition 3. The set of equations (3.15a)(3.15b)(3.15c) is equivalent to the set of equations

\[
\frac{\partial^2 F'}{\partial \varpi^2} + \frac{\partial^2 F'}{\partial z^2} + \frac{1}{\Pi} \left( \frac{\partial F'}{\partial \varpi} \frac{\partial \Pi}{\partial \varpi} + \frac{\partial F'}{\partial z} \frac{\partial \Pi}{\partial z} \right) + \frac{e^{4F'}}{2\Pi^2} \left[ \left( \frac{\partial A'}{\partial \varpi} \right)^2 + \left( \frac{\partial A'}{\partial z} \right)^2 \right] = \frac{4\pi G}{c^4} e^{-2F' + 2K'} (c^2 \rho + 3P), \tag{3.16a}
\]

\[
\frac{\partial}{\partial \varpi} \left( \frac{e^{4F'}}{\Pi} \frac{\partial A'}{\partial \varpi} \right) + \frac{\partial}{\partial z} \left( \frac{e^{4F'}}{\Pi} \frac{\partial A'}{\partial z} \right) = 0, \tag{3.16b}
\]

\[
\frac{\partial^2 \Pi}{\partial \varpi^2} + \frac{\partial^2 \Pi}{\partial z^2} = \frac{16\pi G}{c^4} e^{-2F' + 2K'} P \Pi. \tag{3.16c}
\]
Remark 1. If we consider the dust for which $P = 0$, then (3.16c) says that $\Pi(\varpi, z)$ is a harmonic function and we can assume $\Pi = \varpi$ by conformal change of coordinates. This was first used by [Weyl]* and generalized to the present case by [Lewis].) But it is not the case when $P \neq 0$.

Proposition 4. The set of equations (3.15d)(3.15e) is equivalent to
the set of equations

\[
\frac{\partial \Pi}{\partial \boldsymbol{\varpi}} \frac{\partial K'}{\partial \boldsymbol{\varpi}} - \frac{\partial \Pi}{\partial z} \frac{\partial K'}{\partial z} = \frac{1}{2} \left( \frac{\partial^2 \Pi}{\partial \boldsymbol{\varpi}^2} - \frac{\partial^2 \Pi}{\partial z^2} \right) + \Pi \left[ \left( \frac{\partial F'}{\partial \boldsymbol{\varpi}} \right)^2 - \left( \frac{\partial F'}{\partial z} \right)^2 \right] + \\
- \frac{e^{4F'}}{4\Pi} \left[ \left( \frac{\partial A'}{\partial \boldsymbol{\varpi}} \right)^2 - \left( \frac{\partial A'}{\partial z} \right)^2 \right], \quad (3.17a)
\]

\[
\frac{\partial \Pi}{\partial z} \frac{\partial K'}{\partial \boldsymbol{\varpi}} + \frac{\partial \Pi}{\partial \boldsymbol{\varpi}} \frac{\partial K'}{\partial z} = \frac{\partial^2 \Pi}{\partial \boldsymbol{\varpi} \partial z} + 2\Pi \frac{\partial F'}{\partial \boldsymbol{\varpi}} \frac{\partial F'}{\partial z} - \frac{e^{4F'}}{2\Pi} \frac{\partial A'}{\partial \boldsymbol{\varpi}} \frac{\partial A'}{\partial z}, \quad (3.17b)
\]

provided that the assumption

\[(C2): \left( \frac{\partial \Pi}{\partial \boldsymbol{\varpi}} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \text{ does not vanish.}\]

holds
Proof. Suppose that (3.17a),(3.17b) hold. Then we have $R_{11} = R_{33}$ together with $R_{13} = 0$. Let us consider

$$Q^{\mu\nu} = R^{\mu\nu} - \frac{8\pi G}{c^4} \Phi^{\mu\nu}.$$ 

We already know that $Q^{\mu\nu} = 0$ for $(\mu, \nu) \neq (1, 1), (3, 3)$ and $Q^{11} = Q^{33}$. We want to show that $Q := Q^{11} = Q^{33}$ vanishes.

Since the Euler equations $\nabla_\mu T^{\mu\nu} = 0$ and the Bianchi identities

$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$ hold, we have

$$\nabla_\mu Q^{\mu j} = \frac{1}{2} g^{jj} \partial_j \left( R + \frac{8\pi G}{c^4} T \right) = -\frac{1}{2} e^{-m} \partial_j \left( R + \frac{8\pi G}{c^4} T \right), \quad j = 1, 3.$$ 

But

$$\nabla_\mu Q^{\mu j} = \left[ \partial_j + \left( \partial_j m + \frac{\partial_j \Pi}{\Pi} \right) \right] Q, \quad j = 1, 3,$$
since we already know $Q^{\mu\nu} = 0$ for $(\mu, \nu) \neq (1, 1), (3, 3)$ and

$$\Gamma^\mu_{j\mu} = \partial_j m + \frac{\partial_j \Pi}{\Pi}, \quad \text{and} \quad \Gamma^j_{11} + \Gamma^j_{33} = 0 \quad \text{for} \quad j = 1, 3.$$ 

On the other hand, by contraction we have

$$(g_{11} + g_{33}) Q = -2e^m Q = R + \frac{8\pi G}{c^4} T.$$ 

Therefore

$$\left[ \partial_j + \left( \partial_j m + \frac{\partial_j \Pi}{\Pi} \right) \right] Q = -\frac{1}{2} e^{-m} \partial_j (-2e^m Q) = [\partial_j + (\partial_j m)] Q,$$

for $j = 1, 3$, that is, $\frac{\partial_j \Pi}{\Pi} Q = 0$ for $j = 1, 3$. Under the assumption (C2), it should hold that $Q = Q^{11} = Q^{33} = 0$. So, under this assumption, we can claim that (3.15d), (3.15e) hold. □
Summing up, the system of equations to be solved turns out to be
\[
\frac{\partial^2 F'}{\partial \omega^2} + \frac{\partial^2 F'}{\partial z^2} + \frac{1}{\Pi} \left( \frac{\partial F'}{\partial \omega} \frac{\partial \Pi}{\partial \omega} + \frac{\partial F'}{\partial z} \frac{\partial \Pi}{\partial z} \right) + \frac{e^{4F'}}{2\Pi^2} \left[ \left( \frac{\partial A'}{\partial \omega} \right)^2 + \left( \frac{\partial A'}{\partial z} \right)^2 \right] \\
= \frac{4\pi G}{c^4} e^{2F'+2K'} (c^2 \rho + 3P),
\]

(3.18a)

\[
\frac{\partial}{\partial \omega} \left( \frac{e^{4F'}}{\Pi} \frac{\partial A'}{\partial \omega} \right) + \frac{\partial}{\partial z} \left( \frac{e^{4F'}}{\Pi} \frac{\partial A'}{\partial z} \right) = 0,
\]

(3.18b)

\[
\frac{\partial^2 \Pi}{\partial \omega^2} + \frac{\partial^2 \Pi}{\partial z^2} = \frac{16\pi G}{c^4} e^{2F'+2K'} P\Pi,
\]

(3.18c)

\[
\frac{\partial \Pi}{\partial \omega} \frac{\partial K'}{\partial \omega} - \frac{\partial \Pi}{\partial z} \frac{\partial K'}{\partial z} = \frac{1}{2} \left( \frac{\partial^2 \Pi}{\partial \omega^2} - \frac{\partial^2 \Pi}{\partial z^2} \right) + \Pi \left[ \left( \frac{\partial F'}{\partial \omega} \right)^2 - \left( \frac{\partial F'}{\partial z} \right)^2 \right] + \\
- \frac{e^{4F'}}{4\Pi} \left[ \left( \frac{\partial A'}{\partial \omega} \right)^2 - \left( \frac{\partial A'}{\partial z} \right)^2 \right],
\]

(3.18d)

\[
\frac{\partial \Pi}{\partial z} \frac{\partial K'}{\partial \omega} + \frac{\partial \Pi}{\partial \omega} \frac{\partial K'}{\partial z} = \frac{\partial^2 \Pi}{\partial \omega \partial z} + 2\Pi \frac{\partial F'}{\partial \omega} \frac{\partial F'}{\partial z} - \frac{e^{4F'}}{2\Pi} \frac{\partial A'}{\partial \omega} \frac{\partial A'}{\partial z},
\]

(3.18e)

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and

\[ F' = -\frac{u}{c^2} + \text{Const.} \]  \hspace{1cm} (3.19)

Here \( \rho, P \) in the right-hand sides of (3.18a), (3.18c) should be considered as given functions of \( u \).

See (1.34)(1.35)(1.26) of [Meinel] *16

---

Hereafter we assume \((C2)\). Then the equations (3.18d)(3.18e) can be solved as

\[
\frac{\partial K'}{\partial \varpi} = \left[\left(\frac{\partial \Pi}{\partial \varpi}\right)^2 + \left(\frac{\partial \Pi}{\partial z}\right)^2\right]^{-1} \left(\frac{\partial \Pi}{\partial \varpi} \cdot \text{RH}(3.18d) + \frac{\partial \Pi}{\partial z} \cdot \text{RH}(3.18e)\right),
\]

(3.20a)

\[
\frac{\partial K'}{\partial z} = \left[\left(\frac{\partial \Pi}{\partial \varpi}\right)^2 + \left(\frac{\partial \Pi}{\partial z}\right)^2\right]^{-1} \left(-\frac{\partial \Pi}{\partial z} \cdot \text{RH}(3.18d) + \frac{\partial \Pi}{\partial \varpi} \cdot \text{RH}(3.18e)\right),
\]

(3.20b)

where \(\text{RH}(3.18d), \text{RH}(3.18e)\) stand for the right-hand side of (3.18d), of (3.18e), respectively.
Here we have a serious question: In order that $K'$ which satisfies (3.20a)(3.20b) exists it is necessary that the consistency condition
\[
\frac{\partial}{\partial z}\text{RH}(3.20a) = \frac{\partial}{\partial \varpi}\text{RH}(3.20b).
\] (3.21)
holds. Is it guaranteed? Actually we can verify this consistency if $P = 0$ by taking $\Pi = \varpi$. See [Islam] *17 §2.2, §4.2. But we should be careful when $P \neq 0, \Pi \neq \varpi$.

By direct but tedious calculations, we can get the following observation:

**Proposition 5.** Let $K'$ be arbitrarily fixed and let $F', A', \Pi, \rho$ satisfy (3.18a), (3.18b), (3.18c) and (3.19) with this fixed $K'$. Let us denote by $\tilde{K}_1, \tilde{K}_3$ the right-hand sides of (3.20a), (3.20b), respectively, evaluated by these $F', A', \Pi$. Then it holds that

\[
\frac{\partial \tilde{K}_1}{\partial z} - \frac{\partial \tilde{K}_3}{\partial \varpi} = \frac{8\pi G}{c^4} e^{-2F'+2K'} P\Pi \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \times \\
\times \left[ \left( \frac{\partial K'}{\partial \varpi} - \tilde{K}_1 \right) \frac{\partial \Pi}{\partial z} - \left( \frac{\partial K'}{\partial z} - \tilde{K}_3 \right) \frac{\partial \Pi}{\partial \varpi} \right]. \tag{3.22}
\]
Therefore, as a conclusion, if $K'$ satisfies (3.20a) (3.20b), then the consistency condition (3.21) holds, since

\[
\frac{\partial K'}{\partial \varpi} = \tilde{K}_1, \quad \frac{\partial K'}{\partial z} = \tilde{K}_3.
\]

Of course this conclusion in itself is a vicious circular argument of no use. However the following argument is useful:
Lemma 1. Let us consider a bounded disk $D = \{(\varpi, z) \mid r = \sqrt{\varpi^2 + z^2} < R\}$ of $(\varpi, z)$-plane and denote by $\bar{D}$ the closure of $D$. Suppose that $K' \in C(\bar{D})$ is given and that $F', A', \rho \in C^2(\bar{D}), \Pi \in C^3(\bar{D})$ satisfy (3.18a),(3.18b),(3.18c) and (3.19) with this $K'$. Suppose (B2) holds on $\bar{D}$. Let us denote by $\tilde{K}_1, \tilde{K}_3$ the right-hand sides of (3.20a),(3.20b), respectively, evaluated by these $F', A', \Pi$. (They are $C^1$-functions on $\bar{D}$.) Put

$$\tilde{K}(\varpi, z) := \int_0^z \tilde{K}_3(0, z')dz' + \int_0^{\varpi} \tilde{K}_1(\varpi', z)d\varpi'$$

(3.23)

for $(\varpi, z) \in D$. If $\tilde{K} = K'$, then $K'$ satisfies

$$\frac{\partial K'}{\partial \varpi} = \tilde{K}_1, \quad \frac{\partial K'}{\partial z} = \tilde{K}_3,$$

(3.24)

that is, the equations (3.18d)(3.18e) are satisfied.
**Proof.** Suppose that $\tilde{K} = K'$. It follows from (3.23) with $\tilde{K} = K'$ that

$$
\frac{\partial K'}{\partial \varpi}(\varpi, z) = \tilde{K}_1(\varpi, z),
$$

(3.25)

$$
\frac{\partial K'}{\partial z}(\varpi, z) = \tilde{K}_3(0, z) + \int_0^\varpi \frac{\partial \tilde{K}_1}{\partial z}(\varpi', z) d\varpi'.
$$

(3.26)

Put

$$
L(\varpi, z) := \frac{\partial \tilde{K}_1}{\partial z} - \frac{\partial \tilde{K}_3}{\partial \varpi},
$$

(3.27)

which is a continuous function on $\tilde{\mathcal{D}}$. Then (3.26) reads

$$
\frac{\partial K'}{\partial z}(\varpi, z) = \tilde{K}_3(\varpi, z) + \int_0^{\varpi} L(\varpi', z) d\varpi'.
$$

(3.28)
Now therefore (3.22) of Proposition 5 reads

\[ L(\varpi, z) = -\frac{8\pi G}{c^4} e^{-2F'+2K'} P\Pi \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \frac{\partial \Pi}{\partial \varpi} \int_0^\varpi L(\varpi', z) d\varpi'. \]

(3.29)

Since the function

\[ \frac{8\pi G}{c^4} e^{-2F'+2K'} P\Pi \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \frac{\partial \Pi}{\partial \varpi} \]

is bounded on the compact \( \tilde{D} \) thanks to (B2), the Gronwall’s argument implies that \( L(\varpi, z) = 0 \) on \( \mathcal{D} \) so that (3.28) reads

\[ \frac{\partial K'}{\partial z}(\varpi, z) = \tilde{K}_3(\varpi, z). \]

(3.30)

Thus (3.25) and (3.30) complete the proof. □
3.4 Post-Newtonian approximation

We are going to find a solution of (3.18a)(3.18b)(3.18c)(3.18d)(3.18e)(3.19) which approaches to the solution of the Euler-Poisson equations constructed in [JJTM1], [JJTM2] as $c \to +\infty$. 
Recall that we are assuming Assumption 1 and there are smooth functions $\Upsilon_u, \Upsilon_\rho, \Upsilon_P$ which vanish at 0 such that

$$u := \int_0^\rho \frac{dP}{\rho + P/c^2} = \frac{A\gamma}{\gamma - 1} \rho^{\gamma - 1} (1 + \Upsilon_u (A\rho^{\gamma - 1}/c^2)) \quad \text{for} \quad \rho > 0,$$

(3.31a)

$$\rho = \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma - 1}} u^{\frac{1}{\gamma - 1}} (1 + \Upsilon_\rho (u/c^2)) \quad \text{for} \quad u > 0,$$

(3.31b)

$$P = A^{- \frac{1}{\gamma - 1}} \left( \frac{\gamma - 1}{\gamma} \right)^{\frac{\gamma}{\gamma - 1}} u^{\frac{\gamma}{\gamma - 1}} (1 + \Upsilon_P (u/c^2)) \quad \text{for} \quad u > 0.$$

(3.31c)
Now let us fix the Newtonian limit

\[ u_N = u_O \Theta \left( \frac{r}{a}, \zeta; \frac{1}{\gamma - 1}, b \right), \]  

\[ a = \frac{1}{\sqrt{4\pi G}} \left( \frac{A\gamma}{\gamma - 1} \right)^{\frac{1}{2(\gamma - 1)}} u_O^{-\frac{2-\gamma}{2(\gamma - 1)}}, \quad b = \frac{\Omega^2}{2\pi G} \left( \frac{A\gamma}{\gamma - 1} \right)^{\frac{1}{\gamma - 1}} u_O^{-\frac{1}{\gamma - 1}}, \]  

(3.32b)

where \( u_O = u_N(0) \) is a given positive number,

\[ r = \sqrt{\omega^2 + z^2}, \quad \zeta = \frac{z}{\sqrt{\omega^2 + z^2}}, \]

and \( \Theta \) is the distorted Lane-Emden function of index \( \frac{1}{\gamma - 1} \) with parameter \( b \) which is supposed to be sufficiently small. Therefore we consider
it on the domain

\[ \mathcal{D} = \{(\varpi, z) \mid r = \sqrt{\varpi^2 + z^2} < R\}, \]

in which the support of \( u_N \) is included. Here \( R \) is fixed so that

\[ 2a\xi_1(\frac{1}{\gamma-1}) \leq R \leq a\Xi_0. \]
We consider the perturbation $w$ defined by

$$u = u_N + \frac{w}{c^2},$$  \hspace{1cm} (3.33)

$w$ being an unknown function defined on $\mathcal{D}$ which satisfies

$$w(0, 0) = 0.$$  \hspace{1cm} (3.34)
Of course the Newtonian limits $\rho_N, P_N$ are 

$$\rho_N = \left(\frac{\gamma - 1}{A\gamma}\right)^{1/(\gamma - 1)} (u_N \vee 0)^{1/(\gamma - 1)}, \quad (3.35a)$$

$$P_N = A^{-1/(\gamma - 1)} \left(\frac{\gamma - 1}{\gamma}\right)^{-1/(\gamma - 1)} (u_N \vee 0)^{\gamma/(\gamma - 1)}. \quad (3.35b)$$

Moreover we see

$$c^2(\rho - \rho_N) = \frac{1}{\gamma - 1} \frac{\rho_N}{u_N} w + \Upsilon_1 \rho_N u_N +$$

$$+ c^2 H_\rho(w) + \frac{\gamma_1}{c^2} \left[ \rho_N w + \frac{1}{\gamma - 1} \frac{\rho_N}{u_N} w u + c^2 H_\rho(w) u \right] +$$

$$+ \frac{\gamma_2}{c^2} f_\rho(u) u^2 \left( 1 + \Upsilon''(u/c^2) \right), \quad (3.36)$$

where 

$$u = u_N + \frac{w}{c^2}$$

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and the function $H_\rho(w)$ is defined by

$$H_\rho(w) = f_\rho\left(u_N + \frac{w}{c^2}\right) - f_\rho(u_N) - Df_\rho(u_N)\frac{w}{c^2} \quad (3.37)$$

from the function $f_\rho(u)$, which gives $\rho_N$ from $u_N$,:

$$f_\rho(u) = \left(\frac{\gamma - 1}{A\gamma}\right)^{\frac{1}{\gamma - 1}} (u \vee 0)^{\frac{1}{\gamma - 1}}. \quad (3.38)$$

The constants $\Upsilon_1, \Upsilon_2$ are those appearing in the expression of $\Upsilon_\rho$:

$$\Upsilon_\rho(\xi) = \Upsilon_1 \xi + \Upsilon_2 \xi^2 (1 + \Upsilon''(\xi)), \quad (3.39)$$

where $\Upsilon''(0) = 0$. Here we note that

$$Df_\rho(u_N) = \frac{1}{\gamma - 1} \frac{\rho_N}{u_N}.$$
Let us introduce the variables $V, X, Y$ to put

$$c^2 F' = \Phi_N - \frac{\Omega^2}{2} \varpi^2 - \frac{w}{c^2},$$  \hspace{1cm} (3.40)

$$c^2 K' = -\frac{\Omega^2}{2} \varpi^2 + \frac{V}{c^2},$$  \hspace{1cm} (3.41)

$$\Pi = \varpi \left(1 + \frac{X}{c^4}\right),$$  \hspace{1cm} (3.42)

$$A' = -\frac{\Omega}{c} \varpi^2 \left(1 + \frac{Y}{c^2}\right).$$  \hspace{1cm} (3.43)
Here let us recall

\[ C_N := u_N + \Phi_N - \frac{\Omega^2}{2} \omega^2 = \text{Const.} \]  \hspace{1cm} (3.44)

Therefore, putting

\[ \Phi_N' := \Phi_N - \frac{\Omega^2}{2} \omega^2, \]  \hspace{1cm} (3.45)

we see that (3.40) means to put

\[ \Phi' := c^2 F' = -u + C_N = \Phi_N' - \frac{w}{c^2} = \Phi_N - \frac{\Omega^2}{2} \omega^2 - \frac{w}{c^2}. \]  \hspace{1cm} (3.46)
We are going to rewrite the equations for the unknown variables $w, V, X, Y$.

The equations (3.18a)(3.18b)(3.18c) are reduced to

$$\frac{\partial^2 w}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial w}{\partial \varpi} + \frac{\partial^2 w}{\partial z^2} + \frac{4\pi G}{\gamma - 1} \frac{\rho_N}{u_N} w - 8\Phi'_N \Omega^2 - 2\Omega^2 Y_1 +$$

$$+ 4\pi G \left( -2\Phi_N \rho_N + \Upsilon_1 \rho_N u_N + 3P \right) + R_a = 0 \quad \text{with} \quad w(0, 0) = 0,$$

(3.47a)

$$\frac{\partial^2 Y}{\partial \varpi^2} + \frac{3}{\varpi} \frac{\partial Y}{\partial \varpi} + \frac{\partial^2 Y}{\partial z^2} + \frac{8}{\varpi} \frac{\partial \Phi'_N}{\partial \varpi} + R_b = 0,$$

(3.47b)

$$\frac{\partial^2 X}{\partial \varpi^2} + \frac{2}{\varpi} \frac{\partial X}{\partial \varpi} + \frac{\partial^2 X}{\partial z^2} - 16\pi G P_N + R_c = 0,$$

(3.47c)

where $Y_1 = 2Y + \varpi \frac{\partial Y}{\partial \varpi}$ and $\Upsilon_1$ is the constant in (3.39).
The terms $R_a, R_b, R_c$ are as following:
\[
R_a := - \left( 1 + \frac{X}{c^4} \right)^{-1} \frac{1}{c^2} \frac{\partial X}{\partial \varpi} \frac{1}{\varpi} \frac{\partial \Phi'}{\partial \varpi} \\
- \left[ \left( 1 + \frac{X}{c^4} \right)^{-1} e^{4F'} - 1 - \frac{4\Phi_N'}{c^2} \right] \cdot 2c^2 \Omega^2 \\
- \left[ \left( 1 + \frac{X}{c^4} \right)^{-1} e^{4F'} - 1 \right] \cdot 2\Omega^2 Y_1 \\
- \left( 1 + \frac{X}{c^4} \right)^{-1} e^{4F'} \cdot \frac{\Omega^2}{2c^2} \left[ (Y_1)^2 + \varpi^2 (Y_3)^2 \right], \\
+ 4\pi G \left[ c^2 (\rho - \rho_N) - \frac{1}{\gamma - 1} \frac{\rho_N}{u_N} w - \Upsilon_1 \rho_N u_N \\
+ c^2 (e^{2(K'-F')} - 1)(\rho - \rho_N) + c^2 \left( e^{2(K'-F')} - 1 + \frac{2\Phi_N}{c^2} \right) \rho_N \\
+ 3(e^{2(K'-F')} - 1)P + 3(P - P_N) \right].
\]

(3.48)
\[
R_b := -\frac{8}{c^2} \frac{1}{\omega} \frac{\partial w}{\partial \omega} - \left(1 + \frac{X}{c^4}\right)^{-1} \frac{1}{c^2} \frac{1}{\omega} \frac{\partial X}{\partial \omega} \left(2 + \frac{Y_1}{c^2}\right) + \frac{1}{c^4} \left(1 + \frac{X}{c^4}\right)^{-1} \frac{\partial X}{\partial z} Y_3 \\
+ \frac{4}{c^2} \frac{1}{\omega} \frac{\partial \Phi'}{\partial \omega} \cdot Y_1 + \frac{4}{c^2} \frac{\partial \Phi'}{\partial z} \cdot Y_3, \quad (3.49)
\]

\[
R_c := 16\pi G \left[ - e^{2(K' - F')} P + P_N - \frac{1}{c^4} e^{2(K' - F')} PX \right]. \quad (3.50)
\]

Here we should read

\[
c^2 F' = \Phi' = \Phi_N - \frac{w}{c^2} = -\frac{\Omega^2}{2} \omega^2 + \Phi_N - \frac{w}{c^2}, \quad (3.51)
\]

\[
c^2 (K' - F') = -\frac{\Omega^2}{2} \omega^2 - \Phi'_N + \frac{1}{c^2} (V + w) = -\Phi_N + \frac{1}{c^2} (V + w). \quad (3.52)
\]

Note that we have used the identity

\[
\left(1 + \frac{X}{c^4}\right)^{-1} \left(1 + \frac{X_1}{c^4}\right) - 1 = \left(1 + \frac{X}{c^4}\right)^{-1} \frac{1}{c^4} \omega \frac{\partial X}{\partial \omega}, \quad (3.53)
\]
while

\[ X_1 := X + \omega \frac{\partial X}{\partial \omega} \]

gives

\[ \frac{\partial \Pi}{\partial \omega} = 1 + \frac{X_1}{c^4}. \]

We have denoted

\[ Y_1 = 2Y + \omega \frac{\partial Y}{\partial \omega}, \quad Y_3 = \frac{\partial Y}{\partial z}. \] (3.54)
Now the equations for $V$ are

\[
\frac{\partial V}{\partial \varpi} = c^2 \Omega^2 \varpi + c^4 \cdot \text{RH}(3.20a), \quad (3.55a) \\
\frac{\partial V}{\partial z} = c^4 \cdot \text{RH}(3.20b).
\]

These read

\[
\frac{\partial V}{\partial \varpi} = -4\Phi'_N \varpi \Omega^2 + \frac{\partial X}{\partial \varpi} + \frac{\varpi}{2} \left( \frac{\partial^2 X}{\partial \varpi^2} - \frac{\partial^2 X}{\partial z^2} \right) + \\
+ \varpi \left( \left( \frac{\partial \Phi'_N}{\partial \varpi} \right)^2 - \left( \frac{\partial \Phi'_N}{\partial z} \right)^2 \right) + \varpi \Omega Y_1 + R_d,
\]

\[
\frac{\partial V}{\partial z} = \frac{\partial X}{\partial z} + \varpi \frac{\partial^2 X}{\partial \varpi \partial z} + 2\varpi \frac{\partial \Phi'_N}{\partial \varpi} \frac{\partial \Phi'_N}{\partial z} + \varpi^2 \Omega Y_3 + R_e.
\]

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where

\[
R_d = c^2 \Omega^2 \varpi \left[ 1 - \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) \left( 1 + \frac{X}{c^4} \right)^{-1} e^{4F'} + \frac{4\Phi'_N}{c^2} \right]
+ \frac{1}{2} \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) - 1 \right] \left( \frac{\partial X}{\partial \varpi} + \varpi \frac{\partial^2 X}{\partial \varpi^2} - \varpi \frac{\partial^2 X}{\partial z^2} \right)
+ \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) \left( 1 + \frac{X_1}{c^4} \right) - 1 \right] \varpi \left[ \left( \frac{\partial \Phi'_N}{\partial \varpi} \right)^2 - \left( \frac{\partial \Phi'_N}{\partial z} \right)^2 \right]
- \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) \left( 1 + \frac{X_1}{c^4} \right) e^{4F'} - 1 \right] \varpi \Omega^2 Y_1
+ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \varpi \frac{\partial X}{c^2 \partial z} \left[ \frac{1}{c^2} \left( \frac{\partial X}{\partial z} + \varpi \frac{\partial^2 X}{\partial \varpi \partial z} \right) \right]
+ \frac{2\varpi}{c^2} \left( 1 + \frac{X}{c^4} \right) \frac{\partial \Phi'}{\partial \varpi} \frac{\partial \Phi'}{\partial z}
- \frac{e^{4F'}}{2} \left( 1 + \frac{X}{c^4} \right)^{-1} \varpi^2 \left( -2\Omega + \frac{\Omega^2}{c^2} Y_1 Y_3 \right),
\]

(3.57)
\[ R_e = \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) - 1 \right] \left( \frac{\partial X}{\partial \varpi} + \varpi \frac{\partial^2 X}{\partial \varpi \partial z} \right) \\
+ \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) \left( 1 + \frac{X}{c^4} \right) - 1 \right] 2\varpi \frac{\partial \Phi_N}{\partial \varpi} \frac{\partial \Phi_N}{\partial z} \\
- \left[ \left( 1 + \frac{X_1}{c^4} \right)^{-1} \left( 1 + \frac{X_1}{c^4} \right) \left( 1 + \frac{X}{c^4} \right)^{-1} e^{4F'} - 1 \right] \Omega^2 \varpi^2 Y_3 \\
- \left( 1 + \frac{X_1}{c^4} \right)^{-1} \varpi \frac{\partial X}{c^2 \partial z} \left[ \frac{1}{2c^2} \left( 2 \frac{\partial X}{\partial \varpi} + \varpi \frac{\partial^2 X}{\partial \varpi^2} - \varpi \frac{\partial^2 X}{\partial z^2} \right) \\
+ \frac{\varpi}{c^2} \left( 1 + \frac{X}{c^4} \right) \frac{\partial \Phi'}{\partial \varpi} \frac{\partial \Phi'}{\partial z} \\
- \frac{e^{4F'}}{4} \left( 1 + \frac{X}{c^4} \right)^{-1} \varpi \left( \Omega^2 \left( 2 + \frac{Y_1}{c^2} \right)^2 - \frac{\varpi^2}{c^4} \Omega^2 Y_3^2 \right) \right]. \quad (3.58) \]
Here we put

\[ X_* := 2 \left( X + \varpi \frac{\partial X}{\partial \varpi} \right) + \frac{1}{c^4} \left[ X^2 + 2X \varpi \frac{\partial X}{\partial \varpi} + \varpi^2 \left( \left( \frac{\partial X}{\partial \varpi} \right)^2 + \left( \frac{\partial X}{\partial z} \right)^2 \right) \right] \]  

(3.59)

so that

\[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 = 1 + \frac{X_*}{c^4}. \]  

(3.60)
Of course, we can restate Lemma 1 as

**Lemma 2.** Let \( \mathcal{D}, \bar{\mathcal{D}} \) be those of Lemma 1. Suppose that \( V \in C(\mathcal{D}) \) is given and that \( Y, w \in C^2(\bar{\mathcal{D}}), X \in C^3(\bar{\mathcal{D}}) \) satisfy (3.47a),(3.47b),(3.47c) with this \( V \). Let us denote by \( \tilde{V}_1, \tilde{V}_3 \) the right-hand sides of (3.56a),(3.56b), respectively, evaluated by these \( w, Y, X \). Put

\[
\tilde{V} := \int_0^\tilde{z} \tilde{V}_3(0, z')dz' + \int_0^{\bar{\omega}} \tilde{V}_1(\bar{\omega}', z)d\bar{\omega}'.
\]  

(3.61)

If \( \tilde{V} = V \), then \( V \) satisfies

\[
\frac{\partial V}{\partial \bar{\omega}} = \tilde{V}_1, \quad \frac{\partial V}{\partial z} = \tilde{V}_3,
\]

(3.62)

that is, the equations (3.56a),(3.56b) are satisfied.
3.5 Functional spaces

We are going to prepare some notations to prove the existence of solutions.

3.5.1

Let $n = 3, 4, 5$. For given $\Xi > 0$, we denote

$$B^{(n)}(\Xi) = \{ \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n \mid |\xi| = \sqrt{\sum_j (\xi_j)^2} < \Xi \}$$

and $\bar{B}^{(n)}(\Xi) = \{ |\xi| \leq \Xi \}$. 
For a continuous function $f$ on $\bar{B}^{(n)}(\Xi)$ and $l = 0, 1, 2$, we put

$$
\|f; C^l(\bar{B}^{(n)}(\Xi))\| := \sum_{|L| \leq l} \sup_{|\xi| \leq \Xi} |\partial^L \xi f(\xi)|,
$$

where

$$
\partial^L \xi = \left( \frac{\partial}{\partial \xi_1} \right)^{L_1} \cdots \left( \frac{\partial}{\partial \xi_n} \right)^{L_n}
$$

for $L = (L_1, \cdots, L_n)$ and $|L| = L_1 + \cdots + L_n$. Then $\| \cdot ; C^l(\bar{B}^{(n)}(\Xi))\|$ is the usual norm of the Banach space $C^l(\bar{B}^{(n)}(\Xi))$.

Let us fix a number $\alpha$ such that

$$
0 < \alpha < \min\left\{ \frac{1}{\gamma - 1} - 1, 1 \right\}.
$$

(3.63)
We put

\[ \| f; C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \| := \| f; C^{l}(\bar{B}^{(n)}(\Xi)) \| + \]

\[ + \sup_{|\xi'|, |\xi| \leq \Xi, |L| = l} \frac{|\partial^L f(\xi') - \partial^L f(\xi)|}{|\xi' - \xi|^\alpha} \]

and

\[ C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) = \{ f \in C^{l}(\bar{B}^{(n)}(\Xi)) \mid \| f; C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \| < \infty \}. \]

Then we have the following

**Proposition 6.** i) The bilinear mapping \((f, g) \mapsto f \cdot g\) is continuous as

\[ C^{l}(\bar{B}^{(n)}(\Xi)) \times C^{l'}(\bar{B}^{(n)}(\Xi)) \to C^{l \land l'}(\bar{B}^{(n)}(\Xi)) \]
and as

\[ C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \times C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \to C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \]

\[ II) \text{ If } l \leq l', \text{ then the imbedding } C^{l'}(\bar{B}^{(n)}(\Xi)) \to C^{l}(\bar{B}^{(n)}(\Xi)) \text{ and } C^{l',\alpha}(\bar{B}^{(n)}(\Xi)) \to C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \text{ are continuous.} \]

\[ III) \text{ The imbedding } C^{l+1}(\bar{B}^{(n)}(\Xi)) \to C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \text{ is continuous.} \]

\[ IV) \text{ The imbedding } C^{l,\alpha}(\bar{B}^{(n)}(\Xi)) \to C^{l}(\bar{B}^{(n)}(\Xi)) \text{ is compact.} \]

Let us fix \( \Xi \) and \( \Xi_0 \) such that

\[ 2\xi_1\left(\frac{1}{\gamma-1}\right) \leq \Xi < \Xi_0 \]

and a cut off function \( \chi \in C^\infty([0, +\infty[) \) such that \( \chi(\eta) = 1 \) for \( 0 \leq \eta \leq \Xi \), \( 0 < \chi(\eta) < 1 \) for \( \Xi < \eta < (\Xi + \Xi_0)/2 \), and \( \chi(\eta) = 0 \) for \( (\Xi + \Xi_0)/2 \leq \eta \).
For $g \in C^0(\tilde{B}^{(n)}(\Xi_0))$ we put

$$K^{(n)} g(\xi) = \frac{1}{S_n} \int g(\xi') \chi(|\xi'|) \frac{d\xi'}{|\xi - \xi'|^{n-2}} d\xi',$$

(3.64)

with $S_n = 2(n - 2)\pi^{n/2}/\Gamma(n/2))$. Then we have the following

**Proposition 7.** i) The linear operator $K^{(n)}$ is continuous as

$$C^0(\tilde{B}^{(n)}(\Xi_0)) \rightarrow C^1(\tilde{B}^{(n)}(\Xi_0))$$

and

$$C^{0,\alpha}(\tilde{B}^{(n)}(\Xi_0)) \rightarrow C^{2,\alpha}(\tilde{B}^{(n)}(\Xi_0))$$

ii) If $g \in C^{0,\alpha}(\tilde{B}^{(n)}(\Xi_0))$, then $f = K^{(n)} g \in C^{2,\alpha}(\tilde{B}^{(n)}(\Xi_0))$ satisfies

$$\Delta f + g = 0$$
on $B^{(n)}(\Xi))$. Here $\triangle$ denotes the $n$-dimensional Laplace operator

$$\triangle = \triangle^{(n)} = \sum_{j=1}^{n} \left( \frac{\partial}{\partial \xi_j} \right)^2.$$

See, e.g., Theorem 4.5 of [GilbergT]*18.

3.5.2

Let us use the positive parameter \( a \).

For a function \( Q \) of \((\varpi, x)\) and \( n = 3, 4, 5 \), we consider the function \( Q^{b(n)} \) of \( \xi \in \mathbb{R}^n \) defined by

\[
Q^{b(n)}(\xi) = Q(\varpi, z)
\]

with

\[
\xi = \frac{x}{a}, \quad \varpi = \sqrt{(x_1)^2 + \cdots (x_{n-1})^2}, \quad z = x_n.
\]

Considering functions on the region

\[
\bar{\mathcal{D}}(R) := \{(\varpi, x) \mid r = \sqrt{\varpi^2 + z^2} \leq R\},
\]

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we put

\[ C_l(R) := \{ Q \in C(\overline{D}(R)) \mid Q(\varpi, -z) = Q(\varpi, z) \quad \forall z \} \]

and \[ Q^b(n) \in C^l(\overline{B}^{(n)}(R/a)) \],

\[ \| Q; C_l(R) \| := \| Q^b(n); C^l(\overline{B}^{(n)}(R/a)) \| , \]

\[ C^{l,\alpha}(R) := \{ Q \in C(\overline{D}(R)) \mid Q(\varpi, -z) = Q(\varpi, z) \quad \forall z \} \]

and \[ Q^b(n) \in C^{l,\alpha}(\overline{B}^{(n)}(R/a)) \}, \]

\[ \| Q; C^{l,\alpha}(R) \| := \| Q^b(n); C^{l,\alpha}(\overline{B}^{(n)}(R/a)) \| \]

It is easy to see that the spaces \( C_l(R), C^{l,\alpha}(R), \) and the norms \( \| \cdot; C_l(R) \|, \| \cdot; C^{l,\alpha} \| \) do not depend on the choice of \( n \), but depend only on \( a \).

Proposition 6 reads
Proposition 8. i) The bilinear mapping \((f, g) \mapsto f \cdot g\) is continuous as
\[
\mathcal{C}^l(R) \times \mathcal{C}^{l'}(R) \to \mathcal{C}^{l \land l'}(R)
\]
and as
\[
\mathcal{C}^{l,\alpha}(R) \times \mathcal{C}^{l,\alpha}(R) \to \mathcal{C}^{l,\alpha}(R)
\]
ii) If \(l \leq l'\), then the imbedding \(\mathcal{C}^{l'}(R) \to \mathcal{C}^l(R)\) and \(\mathcal{C}^{l',\alpha}(R) \to \mathcal{C}^{l,\alpha}(R)\) are continuous.
iii) The imbedding \(\mathcal{C}^{l+1}(R) \to \mathcal{C}^{l,\alpha}(R)\) is continuous.
iv) The imbedding \(\mathcal{C}^{l,\alpha}(R) \to \mathcal{C}^l(R)\) is compact.
Let us fix $R$ such that
\[ 2\xi_1\left(\frac{1}{\gamma - 1}\right) \leq \Xi = \frac{R}{a} < \Xi_0 \]
and put $R_0 = a\Xi_0$.

We define the operator $K^{(n)}$ by
\[
(K^{(n)}Q)^{b(n)} = K^{(n)}Q^{b(n)}.
\]

The operator is well-defined, and turns out to be a compact linear operator in $\mathcal{C}^0(R_0)$. Proposition 7 reads

**Proposition 9.**  

i) The operator $K^{(n)}$ is continuous from $\mathcal{C}^0(R_0)$ into $\mathcal{C}^1(R_0)$, and from $\mathcal{C}^{0,\alpha}(R_0)$ into $\mathcal{C}^{2,\alpha}(R_0)$, these operator norms being independent of $a$. 

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ii) If \( g \in C^{0,\alpha}(R_0) \), then \( Q = a^2 \tilde{K}^{(n)} g \) satisfies the equation

\[
\frac{\partial^2 Q}{\partial \omega^2} + \frac{n-2}{\omega} \frac{\partial Q}{\partial \omega} + \frac{\partial^2 Q}{\partial z^2} + g = 0
\]

on \( \mathcal{D}(R) = \{(\omega, z) | r = \sqrt{\omega^2 + z^2} < R\} \).
3.5.3

**Proposition 10.** There is a bounded linear operator $\mathcal{L}$ in $\mathcal{C}^0(R_0)$ which enjoys the following properties:

i) $\mathcal{L}$ is continuous from $\mathcal{C}^0(R_0)$ into $\mathcal{C}^1(R_0)$, and from $\mathcal{C}^{0,\alpha}(R_0)$ into $\mathcal{C}^{2,\alpha}(R_0)$, the operator norms being independent of $a$.

ii) For $g \in \mathcal{C}^{0,\alpha}(R_0)$, the function $Q = a^2 \mathcal{L}g$ satisfies the equation

$$
\frac{\partial^2 Q}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial Q}{\partial \varpi} + \frac{\partial^2 Q}{\partial z^2} + \frac{4\pi G \rho_N}{\gamma - 1 u_N} Q + g = 0
$$

in $\mathcal{D}(R)$ and $Q(0,0) = 0$.

**Proof.** First we note that

$$
4\pi G \frac{\rho_N}{u_N} = a^{-2} (\Theta \vee 0) \frac{1}{\gamma - 1} - 1.
$$
Here $\Theta$ is the distorted Lane-Emden function $\Theta(r/a, \zeta; \frac{1}{\gamma - 1}, b)$. Thus the equation to be solved for $W = Q^b(3), G = a^2 g^b(3)$ is

$$\Delta W + \frac{1}{\gamma - 1} (\Theta^b \vee 0) \frac{1}{\gamma - 1}^{-1} W + G = 0$$

and we require $W(O) = 0$. This problem reduces to the integral equation

$$W = K \left[ \frac{1}{\gamma - 1} (\Theta^b \vee 0) \frac{1}{\gamma - 1}^{-1} W + G \right],$$

where

$$Kf = K^{(3)} f - (K^{(3)} f)(O).$$

In other words, we define $\mathcal{K}$ by

$$(\mathcal{K}F)^b(3) = K F^b(3) = K^{(3)} F^b(3) - K^{(3)} F^b(3)(O).$$
The integral equation reads

$$Q = \mathcal{K}\left[\frac{1}{\gamma - 1}(\Theta \vee 0)\frac{1}{\gamma - 1} - 1 Q + a^2 g\right].$$

Since the operator

$$\mathcal{I} = I - \mathcal{K}\left[\frac{1}{\gamma - 1}(\Theta \vee 0)\frac{1}{\gamma - 1} - 1 \right] : Q \mapsto Q - \mathcal{K}\left[\frac{1}{\gamma - 1}(\Theta \vee 0)\frac{1}{\gamma - 1} - 1 Q\right]$$

maps $\mathcal{C}^0(R_0)$ into $\mathcal{C}^1(R_0)$, it is a compact operator in $\mathcal{C}^0(R_0)$. But it is known that the kernel of $\mathcal{I}$ is $\{0\}$, that is,

$$h = \mathcal{K}\left[\frac{1}{\gamma - 1}(\Theta \vee 0)\frac{1}{\gamma - 1} - 1 h\right], \quad h \in \mathcal{C}^0(R_0) \quad \Rightarrow h = 0,$$

provided that $b \leq \varepsilon_0$ is sufficiently small. See [JJTM1] and [JJTM2]. Therefore the inverse $\mathcal{I}^{-1}$ turns out to be a bounded linear operator in
$C^0(R_0)$. Then $\mathcal{L} = \mathcal{T}^{-1}\mathcal{R}$ is the required operator. Actually, if $g \in C^0$ and $Q = a^2\mathcal{L}g$, then

$$Q = \mathcal{R}\left[\frac{1}{\gamma - 1}(\Theta \vee 0)^{\frac{1}{\gamma - 1}}Q + a^2g\right]$$

holds so that $Q \in C^1$. Moreover, if $g \in C^{0,\alpha}$, then $\frac{1}{\gamma - 1}(\Theta \vee 0)^{\frac{1}{\gamma - 1}}Q \in C^{0,\alpha}$ implies $Q \in C^{2,\alpha}$, and $Q$ satisfies the equation

$$a^2\left[\frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2}\right]Q + \frac{1}{\gamma - 1}(\Theta \vee 0)^{\frac{1}{\gamma - 1}}Q + a^2g = 0.$$ 

Clearly

$$\|Q; C^1\| \leq C a^2 \|g; C^0\|,$$

$$\|Q; C^{2,\alpha}\| \leq C a^2 \|g; C^{0,\alpha}\|$$

with a constant $C$ independent of $a$. □.
3.6 Existence of solutions

We are going to find solutions $w, Y, X, V$ of (3.47a)(3.47b)(3.47c)(3.56a)(3.56b).

We have fixed a small positive number $\varepsilon_0$ and suppose that $b \leq \varepsilon_0$.
Let us fix $R$ such that

$$2\xi_1\left(\frac{1}{\gamma - 1}\right) \leq \Xi = \frac{R}{a} < \Xi_0 = \frac{R_0}{a}. \quad (3.65)$$

We suppose that with a fixed finite constant $C_0(\geq 1)$ it holds that

$$u_0 \leq C_0, \quad \frac{1}{c_0^2} \leq C_0, \quad \frac{1}{C_0} \leq a. \quad (3.66)$$
Moreover, we fix a positive number $\delta_0$ such that

$$\frac{|w|}{c^2} \leq u_0 \delta_0 \Rightarrow u_N + \frac{w}{c^2} < 0 \quad \text{for} \quad 2a\xi_1 \leq r \leq R_0.$$ (3.67)

Actually it is sufficient to take $\delta_0 < -\max_{|\zeta| \leq 1} \Theta(2\xi_1, \zeta)$. Then the support of $u = u_N + \frac{w}{c^2}$ is included in $\mathcal{D}(2a\xi_1)$ as that of $u_N$, provided that $\frac{|w|}{c^2} \leq u_0 \delta_0$.

We are keeping in mind that

$$a^{-2} = 4\pi G \left( \frac{A\gamma}{\gamma - 1} \right)^{-\frac{1}{\gamma - 1}} u_0^{\frac{1}{\gamma - 1} - 1},$$ (3.68)

$$\Omega^2 = 2\pi G b \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma - 1}} u_0^{\frac{1}{\gamma - 1}} = Cba^{-2}u_0 \leq Ca^{-2}u_0,$$ (3.69)
and

\[ \| u_N; \mathcal{E}^{2,\alpha}(R_0) \| \leq C u_O, \quad \| \rho_N; \mathcal{E}^{2,\alpha}(R_0) \| \leq C A^{-\frac{1}{\gamma-1}} u_O^{\frac{1}{\gamma-1}} \leq C' G^{-1} a^{-2} u_O, \]
\[ \| P_N; \mathcal{E}^{2,\alpha}(R_0) \| \leq C A^{-\frac{1}{\gamma-1}} u_O^{\frac{\gamma}{\gamma-1}} \leq C' G^{-1} a^{-2} u_O^2, \]
\[ \| \Phi_N; \mathcal{E}^{2,\alpha}(R_0) \| \leq C u_O, \quad \| \Phi'_N; \mathcal{E}^{2,\alpha}(R_0) \| \leq C u_O. \]  

(3.70)

Here \( C, C' \) stand for constants which depend upon \( \gamma, \alpha \) only.

First, supposing that \( V \in \mathcal{E}^{0,\alpha}(R_0) \) is given, we are going to solve the equations (3.47a)(3.47b)(3.47c) for unknown \( w, Y, X \) by solving the
integral equations

\[ w = \mathcal{L}(g_a + 2\Omega^2 Y_1 + R_a), \quad (3.71a) \]
\[ Y = \hat{\mathcal{R}}^{(5)}(g_b + R_b), \quad (3.71b) \]
\[ X = \hat{\mathcal{R}}^{(4)}(g_c + R_c), \quad (3.71c) \]

where

\[ g_a = -8\Phi'_N\Omega^2 + 4\pi G(-2\Phi_N u_N + \Upsilon_1 \rho_N u_N + 3P), \quad (3.72a) \]
\[ g_b = \frac{8}{\omega} \frac{\partial \Phi'_N}{\partial \omega}, \quad (3.72b) \]
\[ g_c = -16\pi G P_N. \quad (3.72c) \]
From (3.70), we see

\[ \|g_a; \mathcal{C}^{0,\alpha}(R_0)\| \leq C a^{-2} u_0^2, \]
\[ \|g_b; \mathcal{C}^{0,\alpha}(R_0)\| \leq C a^{-2} u_0, \]
\[ \|g_c; \mathcal{C}^{0,\alpha}(R_0)\| \leq C a^{-2} u_0, \]

so that

\[ \|\mathcal{L}g_a; \mathcal{C}^{2,\alpha}(R_0)\| \leq C u_0^2, \]
\[ \|\mathcal{K}^{(5)} g_b; \mathcal{C}^{2,\alpha}(R_0)\| \leq C u_0, \]
\[ \|\mathcal{K}^{(4)} g_c; \mathcal{C}^{2,\alpha}(R_0)\| \leq C u_0^2. \]
Given $w, Y, X \in C^{0, \alpha}(R_0)$, we evaluate $g_a, R_a, R_b, R_c$ by them, and we put

$$\tilde{w} = \mathcal{L}(g_a + 2\Omega^2\tilde{Y}_1 + R_a), \quad (3.73a)$$
$$\tilde{Y} = \mathcal{R}^{(5)}(g_b + R_b), \quad (3.73b)$$
$$\tilde{X} = \mathcal{R}^{(4)}(g_c + R_c). \quad (3.73c)$$

Here $\tilde{Y}_1$ in the right hand side of (3.73a) means $\varpi \frac{\partial \tilde{Y}}{\partial \varpi} + 2\tilde{Y}$ given by $\tilde{Y}$ determined by (3.73b).

Then our task is to find a fixed set of functions of the mapping $(w, Y, X) \mapsto (\tilde{w}, \tilde{Y}, \tilde{X})$. 

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In order to estimate $R_\alpha$, we shall use the following estimates for the function $H_\rho(w)$ is defined by

$$H_\rho(w) = f_\rho\left(u_N + \frac{w}{c^2}\right) - f_\rho(u_N) - Df_\rho(u_N)\frac{w}{c^2} \quad (3.74)$$

from the function $f_\rho(u)$, which gives $\rho_N$ from $u_N$,

$$f_\rho(u) = \left(\frac{\gamma - 1}{A\gamma}\right)^{\frac{1}{\gamma - 1}} (u \vee 0)^{\frac{1}{\gamma - 1}} : \quad (3.75)$$

**Proposition 11.** i) If $w \in C^0(R_0)$, then

$$\|H_\rho(w); C^0(R_0)\| \leq C\left(\frac{\|w; C^0\|}{c^2}\right)^{\frac{1}{\gamma - 1}}, \quad (3.76)$$

and

$$Gc^2\|H_\rho(w); C^0(R_0)\| \leq Ca^{-2}\left(\frac{1}{c^2}\right)^{\frac{1}{\gamma - 1} - 1} u_0^{-\frac{1}{\gamma - 1} + 1} \|w; C^0\|^{\frac{1}{\gamma - 1}}. \quad (3.77)$$
\[ ii) \text{ If } w_1, w_2 \in \mathcal{C}^0(R_0), \text{ then } \\
\| H_\rho(w_2) - H_\rho(w_1); \mathcal{C}^0(R_0) \| \leq C \left( \frac{1}{c^2} (\| w_1; \mathcal{C}^0 \| + \| w_2; \mathcal{C}^0 \|) \right)^{\frac{1}{\gamma - 1} - 1} \frac{\| w_2 - w_1; \mathcal{C}^0 \|}{c^2}, \] 

\[ (3.78) \]

and

\[ Gc^2 \| H_\rho(w_2) - H_\rho(w_1); \mathcal{C}^0(R_0) \| \leq Ca^{-2} \left( \frac{1}{c^2} \right)^{\frac{1}{\gamma - 1} - 1} u_0^{\frac{1}{\gamma - 1} + 1} (\| w_1; \mathcal{C}^0 \| + \| w_2; \mathcal{C}^0 \|) \] 
\[ \times \| w_2 - w_1; \mathcal{C}^0 \|. \] 

\[ (3.79) \]

\[ iii) \text{ If } w \in \mathcal{C}^1(R_0), \text{ then } H_\rho(w) \in \mathcal{C}^{0,\alpha}(R_0) \text{ and } \\
\| H_\rho(w); \mathcal{C}^{0,\alpha}(R_0) \| \leq Cu_0^{\frac{1}{\gamma - 1} - 1} \left[ \left( 1 + \frac{\| w; \mathcal{C}^1 \|}{c^2 u_0} \right)^{\frac{1}{\gamma - 1} - 1} \frac{\| w; \mathcal{C}^0 \|}{c^2} + \right. \\
\left. + \frac{\| w; \mathcal{C}^0 \| \| w; \mathcal{C}^{0,\alpha} \|}{c^2 u_0} \right], \] 

\[ (3.80) \]
\[
G c^2 \| H_\rho(w); \mathfrak{c}^{0,\alpha}(R_0) \| \leq C a^{-2} \left[ \left(1 + \frac{\|w; \mathfrak{c}^1\|}{c^2 u_O} \right)^{\frac{1}{\gamma-1}-1} \|w; \mathfrak{c}^0\| + \right.
\]
\[
+ \frac{1}{c^2} \frac{\|w; \mathfrak{c}^0\|}{u_O} \|w; \mathfrak{c}^{0,\alpha}\| . \tag{3.81}
\]

iv) If \(w_1, w_2 \in \mathfrak{c}^1(R_0)\), then
\[
\| H_\rho(w_2) - H_\rho(w_1); \mathfrak{c}^{0,\alpha}\| \leq C u_O^{\frac{1}{\gamma-1}-1} \left[ \left(1 + \frac{1}{c^2 u_O} (\|w_1; \mathfrak{c}^1\| + \|w_2; \mathfrak{c}^1\|) \right)^{\frac{1}{\gamma-1}-1}
\]
\[
\|w_2 - w_1; \mathfrak{c}^0\| + \frac{1}{c^2} (\|w_1; \mathfrak{c}^0\| + \|w_2; \mathfrak{c}^0\|) \|w_2 - w_1; \mathfrak{c}^{0,\alpha}\|, \tag{3.82}
\]
and

\[ Gc^2 \| H_\rho(w_2) - H_\rho(w_1); \mathcal{C}^{0,\alpha} \| \leq Ca^{-2} \left[ \left( 1 + \frac{1}{c^2 u_0} (\| w_1; \mathcal{C}^1 \| + \| w_2; \mathcal{C}^1 \|) \right)^{\frac{1}{\gamma-1}} \right] \]

\[ \| w_2 - w_1; \mathcal{C}^0 \| + \frac{1}{c^2 u_0} (\| w_1; \mathcal{C}^0 \| + \| w_2; \mathcal{C}^0 \|) \| w_2 - w_1; \mathcal{C}^{0,\alpha} \|. \]

Proof is elementary, if we use the expression

\[ H_\rho(w) = \int_0^1 \left( Df_\rho \left( u_N + t \frac{w}{c^2} \right) - Df_\rho(u_N) \right) dt \cdot \frac{w}{c^2}, \]

and

\[ H_\rho(w_2) - H_\rho(w_1) = \int_0^1 \left( Df_\rho \left( u_N + \frac{1}{c^2} (w_1 + t(w_2 - w_1)) \right) - Df_\rho(u_N) \right) dt \cdot \frac{w_2 - w_1}{c^2}. \]
Here, of course,

\[ Df_\rho(u) = \frac{1}{\gamma - 1} \left( \frac{\gamma - 1}{A\gamma} \right) \frac{1}{\gamma - 1} (u \vee 0)^{\frac{1}{\gamma - 1} - 1}. \]
Let us fix \( V \in \mathcal{C}^{0, \alpha}(R_0) \) such that \( \| V; \mathcal{C}^{0, \alpha}(R_0) \| \leq u_{O}^{2}M \) and suppose
\[
\frac{u_{O}M}{c^2} \leq C_{0}.
\] (3.84)

Let us consider \( w, Y, X \) such that
\[
\| w; \mathcal{C}^{1} \| \leq u_{O}^{2}B, \quad \| Y; \mathcal{C}^{1} \| \leq u_{O}B, \quad \| X; \mathcal{C}^{1} \| \leq u_{O}^{2}B,
\] (3.85)
and
\[
\| w; \mathcal{C}^{2, \alpha} \| \leq u_{O}^{2}B^{*}, \quad \| Y; \mathcal{C}^{2, \alpha} \| \leq u_{O}B^{*}, \quad \| X; \mathcal{C}^{2, \alpha} \| \leq u_{O}^{2}B^{*},
\] (3.86)
with
\[
B \leq B^{*}.
\] (3.87)
Suppose
\[ \frac{u_Q B}{c^2} \leq \delta_0. \] (3.88)

Using (3.77), we have
\[ \|\tilde{w}; \mathcal{E}^1\| \leq C_1 \left[ u_Q^2 + u_Q \|\tilde{Y}_1; \mathcal{E}^1\| + \frac{u_Q^3}{c^2} (B(1 + B + M + B^*) + M) + \left( \frac{u_Q B}{c^2} \right)^{\gamma-1} u_Q^2 B \right]. \] (3.89)

Here we have used the estimate
\[ \left\| \frac{1}{\omega} \frac{\partial Q}{\partial \omega}; \mathcal{E}^0 \right\| \leq C a^{-2} \|Q; \mathcal{E}^2\|. \]
which can be verified by the identity

\[ a^2 \frac{1}{\varpi} \frac{\partial Q}{\partial \varpi} = \frac{1}{n - 2} \sum_{j=2}^{n-1} \frac{\partial^2 Q^b}{\partial \xi_j^2} \Bigg|_{\xi=(\varpi/a,0,\ldots)}. \]

On the other hand, using (3.81), we have

\[ \| \tilde{w}; \mathcal{E}^{2,\alpha} \| \leq C_1 \left[ u_0^2 + u_0 \| \tilde{Y}; \mathcal{E}^{2,\alpha} \| + \frac{u_0^3}{c^2} (B^*(1 + B^* + M) + M) + u_0^2 B \right]. \]

(3.90)
Clearly we have

\[
\|\tilde{Y}; \mathcal{E}^1\| \leq C_1 \left[ u_0 + \frac{u_0^2}{c^2} (B + B^*) \right], \tag{3.91}
\]

\[
\|\tilde{Y}; \mathcal{E}^{2,\alpha}\| \leq C_1 \left[ u_0 + \frac{u_0^2}{c^2} B^* \right], \tag{3.92}
\]

\[
\|\tilde{X}; \mathcal{E}^1\| \leq \|\tilde{X}; \mathcal{E}^{2,\alpha}\| \leq C_1 \left[ u_0^2 + \frac{u_0^3}{c^2} (B + M) \right]. \tag{3.93}
\]
Then we can find $B, B^*$ such that $B \leq B^*$ which satisfy (3.88) and

$$C_1 \left[ 1 + C_1 \left( 1 + \frac{u_O}{c^2} B^* \right) \right] +$$

$$+ \frac{u_O}{c^2} (B(1 + B + M + B^*) + M) + \left( \frac{u_O B}{c^2} \right)^{\frac{1}{\gamma - 1} - 1} B \right] \leq B,$$  (3.94a)

$$C_1 \left[ 1 + C_1 \left( 1 + \frac{u_O}{c^2} (B + B^*) \right) \right] +$$

$$+ \frac{u_O}{c^2} (B(1 + B + M + B^*) + M) + B \right] \leq B^*, \quad (3.94b)$$

$$C_1 \left[ 1 + \frac{u_O}{c^2} (B + B^*) \right] \leq B, \quad (3.94c)$$

$$C_1 \left[ 1 + \frac{u_O}{c^2} B^* \right] \leq B^*, \quad (3.94d)$$

$$C_1 \left[ 1 + \frac{u_O}{c^2} (B + M) \right] \leq B \leq B^*, \quad (3.94e)$$

provided that the following assumption (D) holds:
(D): \( \frac{u_0}{c^2} \) is sufficiently small, say, \( \frac{u_0}{c^2} \leq \delta_1, \delta_1(\leq 1) \) being a small positive number.

Fixing such a pair \( B, B^* \), we define

**Definition 5.** Put

\[
\mathcal{X} := \{(w, Y, X) \in \mathcal{C}^{2,\alpha}(R_0) \times \mathcal{C}^{2,\alpha}(R_0) \times \mathcal{C}^{2,\alpha}(R_0) \mid (3.85), (3.86) \text{ hold.}\}
\]

(3.95)

Then the mapping \((w, Y, X) \mapsto (\tilde{w}, \tilde{Y}, \tilde{X})\) maps the space \( \mathcal{X} \) into itself.
We are going to show that the mapping \((w, Y, X) \mapsto (\tilde{w}, \tilde{Y}, \tilde{X})\) is a contraction with respect to a suitable norm.

Let us denote \(U = (w, Y, X), \tilde{U} = (\tilde{w}, \tilde{Y}, \tilde{X})\) and so on. We consider the norms

\[
\mathcal{N}(U) := \max\{\|w; \mathcal{E}^1\|, \|u_0 Y; \mathcal{E}^1\|, \|X; \mathcal{E}^1\|\},
\]

\[
\mathcal{N}^*(U) := \max\{\|w; \mathcal{E}^{2,\alpha}\|, \|u_0 Y; \mathcal{E}^{2,\alpha}\|, \|X; \mathcal{E}^{2,\alpha}\|\},
\]

\[
\mathcal{N}(U) := \mathcal{N}(U) + \kappa \mathcal{N}^*(U).
\]

Here \(\kappa\) is a positive constant specified later.

Using (3.79), we see

\[
\mathcal{N}(\tilde{U}_2 - \tilde{U}_1) \leq C_2 \left[ \left(\frac{u_0}{c_2}\right) \alpha \mathcal{N}(U_2 - U_1) + \frac{u_0}{c_2} \mathcal{N}^*(U_2 - U_1) \right].
\]
Using (3.83), we see

\[ \mathcal{N}^*(\tilde{U}_2 - \tilde{U}_1) \leq C_2 \left[ \mathcal{N}(U_2 - N_1) + \frac{uO}{c^2} \mathcal{N}^*(U_2 - U_1) \right]. \quad (3.98) \]

Therefore we see

\[ \mathcal{N}(\tilde{U}_2 - \tilde{U}_1) \leq K[\mathcal{N}(U_2 - U_1) + \kappa' \mathcal{N}^*(U_2 - U_1)], \quad (3.99) \]

where

\[ K = C_2 \left[ \left( \frac{uO}{c^2} \right)^\alpha + \kappa \right], \quad (3.100) \]

\[ \kappa' = \frac{uO}{c^2} (1 + \kappa) \left[ \left( \frac{uO}{c^2} \right)^\alpha + \kappa \right]^{-1}. \quad (3.101) \]

Let us take

\[ \kappa = 2 \left( \frac{uO}{c^2} \right)^{1-\alpha}, \quad (3.102) \]
provided that
\[ 2 \left( \frac{u_0}{c^2} \right)^{1-\alpha} \leq 2\delta_1^{1-\alpha} \leq 1. \] (3.103)

Then we have \( \kappa' \leq \kappa \) so that
\[ \mathcal{N}(\tilde{U}_2 - \tilde{U}_1) \leq K\mathcal{N}(U_2 - U_1). \] (3.104)

But, under the assumption (D) with sufficiently small \( \delta_1 \), we have
\[ K = C_2 \left[ \left( \frac{u_0}{c^2} \right)^\alpha + 2 \left( \frac{u_0}{c^2} \right)^{1-\alpha} \right] \leq C_2 \left[ \delta_1^\alpha + 2\delta_1^{1-\alpha} \right] < 1, \]
say, \( U \mapsto \tilde{U} \) is a contraction with respect to the norm \( \mathcal{N} \). Note that \( \mathcal{N} \) is equivalent to \( \mathcal{N}^* \) since \( \kappa \mathcal{N}^* \leq \mathcal{N} \leq (1 + \kappa)\mathcal{N}^* \).
Summing up, we can claim

**Theorem 4.** There is a unique solution \((w, Y, X)\) of \((3.71a)(3.71b)(3.71c)\) in \(\mathfrak{X}\) for any given \(V \in \mathcal{C}^{0,\alpha}(R_0)\) with \(\|V; \mathcal{C}^{0,\alpha}(R_0)\| \leq u^2_0 M\).

**Definition 6.** We shall denote the solution of Theorem 4 by \(U = (w, Y, X) = S(V)\).

**Theorem 5.** There is a constant \(C\) such that it holds for \(V_\mu \in \mathcal{C}^{0,\alpha}(R_0)\) with \(\|V_\mu; \mathcal{C}^{0,\alpha}(R_0)\| \leq u^2_0 M, \mu = 1, 2,\) that

\[
\mathcal{N}^*(S(V_2) - S(V_1)) \leq C \frac{u_0}{c^2} \|V_2 - V_1; \mathcal{C}^{0,\alpha}(R_0)\|, \tag{3.105}
\]

provided that \(u_0/c^2 \leq \delta_1\) is sufficiently small.
As the second step, we are going to solve the equations (3.56a), (3.56b). More precisely speaking, we are looking for $V \in \mathcal{C}^{0,\alpha}(R_0)$ with $\|V; \mathcal{C}^{0,\alpha}(R_0)\| \leq u_0^2 M$ such that $V, U = S(V)$ satisfy (3.56a), (3.56b).

Note that $V$ does not appear in the right-hand side of (3.56a), (3.56b). Let us denote $T_1(V), T_3(V)$ the right-hand sides of (3.56a), (3.56b), respectively, evaluated by $U = S(V)$, and let us put $\mathcal{T}(V) = \tilde{V}$ defined by

$$\tilde{V} := \int_0^z T_3(V)(0, z')dz' + \int_0^{\varpi} T_1(V)(\varpi', z)d\varpi'. \quad (3.106)$$

Thanks to Lemma 2, if $V$ is a fixed point of the mapping $\mathcal{T}$, it is the required solution on $\mathcal{D}(R)$. 157
We see clearly that there is a constant $C_3$ independent of $M$ such that
\[
\|T_j(V); C^0,\alpha\| \leq C_3 a^{-1} u_0^2, \quad j = 1, 3, \quad (3.107)
\]
for $\|V; C^0,\alpha(R_0)\| \leq u_0^2 M$ and $u_0/c^2 \leq \delta_1 \ll 1$. Then we have
\[
\|T(V); C^0,\alpha\| \leq C_3 u_0^2. \quad (3.108)
\]
So, taking $M$ large so that
\[
C_3 \leq M, \quad (3.109)
\]
we claim that under the mapping $T : V \mapsto \tilde{V}$ the functional set
\[
\mathcal{V} := \{V \in C^0,\alpha(R_0) \mid \|V; C^0,\alpha(R_0)\| \leq u_0^2 M\} \quad (3.110)
\]
is stable. Moreover, thanks to Theorem 5, $T$ is a contraction with respect to the norm $\|\cdot; C^0,\alpha(R_0)\|$, provided $u_0/c^2 \leq \delta_1$ is sufficiently small.

Looking at the right-hand sides of (3.56a),(3.56b), which are of class $C^0,\alpha$, we see $V \in C^1,\alpha$. Then it follows that $X \in C^3,\alpha$, since

$$X = \mathfrak{K}^{(4)}[-16\pi GP_N + R_c]$$

with

$$-16\pi GP_N + R_c = -16\pi Ge^{2(K'-F')}P\left(1 + \frac{X}{c^4}\right) \in C^1,\alpha.$$

In fact $V \in C^1,\alpha$ implies $K' - F' \in C^1,\alpha$ in view of (3.52). As result $V$ turns out to be of class $C^2,\alpha$ in view of (3.56a),(3.56b) again. Thus we have
Theorem 6. There is a solution $V \in \mathcal{C}^{2,\alpha}(R_0)$ of $V = T(V)$ together with $U = S(V)$, provided that $u_O/c^2 \leq \delta_1 \ll 1$. This is the unique solution in $\mathfrak{H}$. The equations (3.56a)(3.56b) are satisfied on $\mathfrak{D}(R)$.

Let us recall

$$F' = -\frac{\Omega^2}{2c^2} \varpi^2 + \frac{\Phi_N}{c^2} - \frac{w}{c^4}, \quad K' = -\frac{\Omega^2}{2c^2} \varpi^2 + \frac{V}{c^4},$$

$$A' = -\frac{\Omega}{c} \varpi \left(1 + \frac{Y}{c^2}\right), \quad \Pi = \varpi \left(1 + \frac{X}{c^4}\right), \quad u = u_N + \frac{w}{c^2}.$$

Note that, if $u_O/c^2 \ll 1$, then the assumption (B1) holds, since

$$e^{2F'} \left(1 - \frac{\Omega}{c} A'\right)^2 - e^{-2F'} \frac{\Omega^2}{c^2} \Pi^2 = 1 + \frac{2\Phi_N}{c^2} + O\left(\frac{u_O^2}{c^4}\right) \quad = 1 + O\left(\frac{u_O}{c^2}\right),$$

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and the assumption (B2) holds, since \( \frac{\partial \Pi}{\partial \varpi} = 1 + O \left( \frac{u_0^2}{c^4} \right) \).

Thus the proof of the main result is complete.
3.7 Discussion

We have constructed an axially symmetric metric on the domain $\mathcal{D} = \{r < R\}$, $R$ being arbitrarily large, such that the support of the density $\overline{\mathcal{R}} = \mathcal{R} \cup \partial \mathcal{R} = \{r \leq r_+(\zeta)\}$ is a compact subset of $\mathcal{D}$. However we have not yet clarified what will happen when we continue the metric as long as possible in the vacuum region for $r \geq R$.

So, the open problem:

**What happens when we prolong the vacuum axisymmetric metric outside $\mathcal{D}$?**

**Can the prolongation be an asymptotic flat one as $r \to +\infty$?**
When we consider a spherically symmetric metric, which is given by the Tolman-Oppenheimer-Volkoff equation, the interior metric can be patched with the Schwarzschild metric on the exterior vacuum region at the boundary $\partial \mathcal{M} = \{ r = \text{Const.} \}$ in $C^2$-manner. See SUPPLEMENTARY REMARK 4 of [ssEE] and Theorem 3 of [TOVdS]. But the author does not know an analogous statement to the Birkhoff’s theorem on spherically symmetric metrics in the case of axially symmetric metrics.

Keeping in mind this spherically symmetric case, and that the Kerr metric is the typical, although not only one, static axially symmetric metric in the vacuum region, we naturally ask:
(Q1): Can we find a Kerr metric described by an appropriate coordinates on the vacuum region \( \{ r_+(\zeta) < r \} \) which is matched to the interior solution constructed on \( \mathcal{R} = \{ \rho > 0 \} = \{ r < r_+(\zeta) \} \) with junction conditions to be expected, say, e.g., with coefficients of class \( C^1 \)?
3.7.1 Matter-Vacuum matching and Ernst equation

An exact formulation of the ‘matter-vacuum matching’ problem to find an asymptotically flat vacuum exterior which can be patched to the interior solution can be found in the short review [MacCallum-MarsVera2004]*19 by three leading experts of this problem, and the

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papers [Mars1999]*20 and [MacCallumMarsVera2007]*21


Actually we note that, if we consider the equations (3.18a) (3.18b) (3.18c) on the vacuum region in which \( \rho = 0 \), the equation (3.18c) admits the particular solution \( \Pi = \varpi \), and then the equations (3.18a), (3.18b) are reduced to

\[
\left[ \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} \right] F' + \frac{e^{4F'}}{\varpi} \left( \left( \frac{\partial A'}{\partial \varpi} \right)^2 + \left( \frac{\partial A'}{\partial z} \right)^2 \right) = 0, \tag{3.111}
\]

\[
\frac{\partial}{\partial \varpi} \left( \frac{e^{4F'}}{\varpi} \frac{\partial A'}{\partial \varpi} \right) + \frac{\partial}{\partial z} \left( \frac{e^{4F'}}{\varpi} \frac{\partial A'}{\partial z} \right) = 0. \tag{3.112}
\]

But it follows from (3.112) that there exists a function \( B'(\varpi, z) \) such that

\[
\frac{\partial B'}{\partial \varpi} = -\frac{e^{4F'}}{\varpi} \frac{\partial A'}{\partial z}, \quad \frac{\partial B'}{\partial z} = \frac{e^{4F'}}{\varpi} \frac{\partial A'}{\partial \varpi}. \tag{3.113}
\]
Then (3.111) reads

\[
\left[ \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} \right] F' + \frac{e^{-4F'}}{2} \left( \left( \frac{\partial B'}{\partial \varpi} \right)^2 + \left( \frac{\partial B'}{\partial z} \right)^2 \right) = 0, \quad (3.114)
\]

and the consistency condition for the existence of \( A' \) reads

\[
\left[ \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} \right] B' - 4 \left( \frac{\partial F'}{\partial \varpi} \frac{\partial B'}{\partial \varpi} + \frac{\partial F'}{\partial z} \frac{\partial B'}{\partial z} \right) = 0. \quad (3.115)
\]

So, the system of equations (3.114)(3.115) governs the ‘Ernst potential’ \( E' = e^{2F'} + \sqrt{-1}B' \). Let us consider \( F, \) and \( B \) which is defined by (3.113) by replacing \( B', A' \) by \( B, A \). Then as remarked in the footnote of p. 9 of [Meinel], \( F, B \) satisfy the same equations (3.114), (3.115) in which \( F', B' \) are replaced by \( F, B \). This system is nothing but (1) [Mars1999] or (1) [MacCallumMarsVera2004], (11) [MacCallum-MarsVera2007]. (\( U, \Omega \) there should read \( F, B \) here.)
The Ernst potential \( E = e^{2F} + \sqrt{-1}B \) obeys the ‘Ernst equation’

\[
\Re[E] \cdot \left[ \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} \right] E = \left( \frac{\partial E}{\partial \varpi} \right)^2 + \left( \frac{\partial E}{\partial z} \right)^2,
\]

which is equivalent to the system (3.114),(3.115). Then the asymptotic flatness condition reads

\[
F = 1 - \frac{GM}{c^2 r} + O\left(\frac{1}{r^2}\right), \quad B = -\frac{2GzJ}{c^3 r^3} + O\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to +\infty
\]

with some constants \( M, J \). Here, of course, \( r = \sqrt{\varpi^2 + z^2} \). See [Mac-CallumMarsVera2004].

Thus it is an important open problem to find an asymptotically flat vacuum metric which is patched to the metric constructed here on an bounded domain.
But we should note that the metric of the form (3.6) with the co-rotating potential $F'$ cannot be global and asymptotically flat. Actually, if it is possible, it should hold at least that

$$ F \to 0, \quad A \to 0, \quad K \to 0, \quad \Pi \sim \varpi $$

as $r \to +\infty$. See p.5, (1.12) [Meinel]. Then the right-hand side of

$$ e^{2F'} = e^{2F} \left(1 + \frac{\Omega}{c} A\right)^2 - e^{-2F} \frac{\Omega^2}{c^2} \Pi^2 \quad \sim \quad 1 - \frac{\Omega^2}{c^2} \varpi^2 $$

turns out to be negative, provided that $\Omega \neq 0$, as $\varpi \to \infty$, and $F'$ cannot be real-valued for $e^{2F'} < 0$. 

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3.7.2 Strategy of U. Heilig

However the formulation of the problem done in the work [Heilig] by U. Heilig is quite different. In this work the basic equations comes from the ‘frame theory’ of Jürgen Ehlers ([Ehlers] *22 ). This formulation of the Einstein’s equations contains the perspective of a post-Newtonian approximation already.

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The basic exposition of this Ehlers’ frame theory can be found in [Lottermoser]*23, while an introductory review can be found in [OliynikShmidt]*24, and recent developments can be found in [Oliynyk2007]*25, [Oliynyk2009]*26, [Oliynyk2010]*27.

U. Heilig [Heilig] claims that the metric is constructed globally, but the result seems not to be a solution of the matter-vacuum matching problem. For the metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is claimed to satisfy

$$g_{\mu\nu} - \eta_{\mu\nu} \in M^p_{2,\delta}(\mathbb{R}^3) \text{ with } p \geq 4, 0 \leq \delta < -2 + 3\frac{p - 1}{p},$$

where $\eta_{\mu\nu} dx^\mu dx^\nu$ is the Minkowski metric, and the functional space $M^p_{2,\delta}(\mathbb{R}^3)$ consists of functions $f$ such that

$$\sum_{|\ell| \leq 2} \| (\sqrt{1 + |x|^2})^{\delta + |\ell|} \partial^{\ell} f \|_{L^p(\mathbb{R}^3)} < \infty.$$

Therefore it seems that this does not guarantees that

$$g_{\mu\nu} - \eta_{\mu\nu} = O\left(\frac{1}{|x|}\right).$$
3.7.3 Strategy of W. Roos

Until now the matter-vacuum matching problem has been discussed under the formulation which is in the opposite direction to the formulation described in the preceding section as \((Q1)\). Namely, the formulation of the problem discussed by several authors until now is:
(Q2): Given an axisymmetric stationary vacuum metric, in particular, the Kerr metric, on the exterior domain $\mathcal{E} = \mathbb{R}^3 \setminus \overline{\mathcal{I}}$ of an interior domain $\mathcal{I}$ with boundary $S = \partial \mathcal{E} = \partial \mathcal{I}$, can we find a solution of the field equations such that $\{\rho > 0\}$ or $\{P > 0\}$ coincides with the interior $\mathcal{I}$ on which the metric is constructed to be matched across $S$ with plausible junction conditions, say, e.g., with coefficients of class $C^1$ to the given exterior vacuum metric?
Several authors have been discussed this problem of so called ‘a source of the Kerr field’. The textbook [PlebanskiKrasinski]*28 by J. Plebański and A. Krasiński, 2006, pp.499-495, says

The Kerr solution has been known for more than 40 years now, and from the very beginning its existence provoked the simple question: what material body could generate such a vacuum field around it? Several authors have tried very hard to find a model of the source, but so far without success. The most promising positive result is that of Roos (1976), who investigated the Einstein equations with a perfect fluid with the boundary condition that the Kerr metric is matched to the solution. All attempts so far to find an explicit example of a solution failed. The continuing lack of success prompted some authors to spread the suspicion that a perfect fluid source might not exist; rumours about this suspicion were then taken as a serious suggestion. The opinion of one of the present authors (A. K.) is that a bright new
idea is needed, as opposed to routine standard tricks tested so far.
However the strategy of W. Roos [Roos]*29 seems to be not so promising.

A rough sketch of the main result of [Roos] giving Roos’ strategy is as follows:

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Fig. 2. (a) "Complete interior coordinate system," which, for instance, exists if the pressure function \( p \) (of \( T_{ab} \)) has nonvanishing gradient except in the "center" where it has its maximum [2]. (b) "Half-local" coordinate system which always exists in a neighborhood of a given boundary surface \( \delta \).
Let be given a stationary axisymmetric $C^\infty$ vacuum field in a domain $\tilde{\mathcal{E}}$, an analytic equation of state $\rho = \rho(P)$, a time-like hypersurface $S$ bounding a simply connected domain $\mathcal{I}$, $\mathcal{E} = \mathbb{R}^3 \setminus \tilde{\mathcal{I}} \subset \tilde{\mathcal{E}}$, such that on $S$ matching conditions and some additional conditions hold. Take a coordinate system $x^1, x^2, x^3$ in a neighborhood $\mathcal{V}$ of $S$ such that $\{x^1 < 0\} \cap \mathcal{V} = \mathcal{E} \cap \mathcal{V}$, or, $\{x^1 > 0\} \cap \mathcal{V} = \mathcal{I} \cap \mathcal{V}$. Then we can find, in a neighborhood of $S$, a unique stationary axisymmetric and analytic field for a rotating perfect fluid satisfying the matching conditions on $S$ by applying the Cauchy-Kovalevskaja theorem on the initial surface $\{x^1 = 0\}$ so that $\partial P/\partial x^1 > 0$ on $x^1 > 0$. 
Let us show a defect of this strategy by applying it to the particular case of spherically symmetric solutions. For the sake of simplicity let us take the geometrical unit system in which $G = c = 1$.

Given the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2)$$

on $\tilde{\mathcal{E}} = \{r > 2M\}$ and $\mathcal{S} = \{r = R\}$, where the data $(R, M)$ belongs to the admissible set $\mathcal{A}$ defined by

$$\mathcal{A} := \{(R, M)| R > 0, M > 0, 1 - \frac{2M}{R} > 0\}.$$

Then the field equations on $\mathcal{I} = \{r < R\}$ is reduced to the Tolman-
Oppenheimer-Volkhoff equation

\[
\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{dP}{dr} = -\left(\rho + P\right) \frac{m + 4\pi r^3 \rho}{r^2(1 - \frac{2m}{r})}. \quad (3.116)
\]

Therefore the application of the Cauchy-Kovalevskaja theorem means the shooting of the solution \((m(r), P(r))\) of (3.116) to the left from \(r = R\) with the initial data \((m, P) = (M, 0)\). Actually the solution exists locally on \([R - \delta, R]\) with \(\delta \ll 1\). The problem is: Can it be prolonged to \(r = 0\) and hit a regular values at the center?

In order to fix the idea, we consider the domain \(D\) of the equation (3.116) as

\[
D = \{0 < r, 0 < P, 0 < m + 4\pi r^3 \rho, 0 < 1 - \frac{2m}{r}\}. \quad (3.117)
\]
(We are assuming that $\rho = \rho(P) > 0$ for $P > 0$.) Let the left maximal existence of existence of the solution in this specified domain $D$ to the initial data $(m, P) = (M, 0)$ be $[r_-, R]$. Note that $dP/dr < 0$ on $D$. The following cases might occur:

Either **Case-(0):** $r_- > 0$, or **Case-(1):** $r_- = 0$.

If **Case-(0),** then either **Case-(00):** $P \nearrow +\infty$ as $r \searrow r_-$, or **Case-(01):** $P \nearrow \exists P_-(< \infty)$ as $r \searrow r_-$.

If **Case-(01),** it can be shown that $dP/dr \to 0$ as $r \to r_- + 0$.

If **Case-(1),** either **Case-(10):** $P \nearrow +\infty$ as $r \searrow 0$, or **Case-(11):** $P \nearrow \exists P_0(< \infty)$ as $r \searrow 0$.

Suppose **Case-(11).** Then

$$m(r) = M - \int_r^R 4\pi \rho(r')(r')^2 dr' \to \exists m_0$$

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as $r \to +0$. Since $m(r) < r/2 \to 0$, we see $m_0 \leq 0$. Since $m(r) + 4\pi r^3 \rho(r) > 0$, we have $m_0 \geq 0$. Therefore $m_0 = 0$ and

$$m(r) = \int_0^r 4\pi \rho(r')(r')^2 dr'.$$

Thus if and only if the Case-(11), the solution is that of (3.116) which is regular at the center. In other words, the Roos’ strategy is successful if and only if Case-(11).

Let us look at the situation in the opposite direction. If we shoot the solution of (3.116) with the initial data $m = 0, P = P_0 > 0$ to the right from $r = +0$, the solution $(m(r), P(r))$ may hit $P = 0$ at finite $r = R$; when we can verify that $M = m(R)$ exists and satisfy $1 - \frac{2M}{R} > 0$. The set $\mathcal{O}$ of all such $P_0$ is an open set of $]0, +\infty[. $
Of course $\mathcal{O}$ can be empty, which depends on the equation of state $\rho = \rho(P)$. The connected components $\mathcal{O}_j, j = 1, 2, \cdots$, are open intervals, and the set of $(R, m(R))$ for $P_0 \in \mathcal{O}_j$ is a curve $\mathcal{C}_j$ in the set of admissible data $\mathcal{A}$. **Case-(11)** is the case and the Roos’ strategy turns out to be successful if and only if $\exists j : (R, M) \in \mathcal{C}_j$. Since the 2-dimensional measure of $\mathcal{C}_j$ is zero, we can say that $(R, M) \notin \bigcup_j \mathcal{C}_j$ a. e. In other words **the strategy of W. Roos is almost everywhere unsuccessful**. In order to make the Roos’ strategy give success we must chose a combination of $M$ and $R$ which fits a very tightly restricted condition, and it seems to be desperate to give an explicit expression of the condition accordingly to the given equation of states.
FIG. 4
3.7.4 Ellipsoid as a candidate for the boundary surface

As for spherically symmetric problem, although it is the case very scarcely, the Roos’ strategy is successful by taking a sphere \( \{r = R\} \) as the matching boundary \( S \). However if the angular velocity \( \Omega \) is not equal to 0, maybe we should take other figures than spheres as \( S \). Inspired by the analogy with the ellipsoidal figures of rotating liquid (that is, incompressible ) stars in the non-relativistic theory (see e.g. [?]), one may try to take ellipsoids as candidates of \( S \) for \((Q2)\).

In fact, P. Collas, [Collas]\(^{30}\) p.68 says

Hernandez outlined a method for constructing exact interior solution which served as sources for the Kerr metric. The guessed metric matches the Kerr metric on a suitable surface [Footnote (5)] and, in the limit of no rotation, goes into the interior Schwarzschild solution.

Footnote (5): A. Krasinski: Institute of Astronomy, Polish Academy of Science preprint No.63, Warsaw (May 1976), has shown that the surface of a source of the Kerr metric should be given by \( r = \text{constant} \) in Boyer-Lindquist co-ordinates.
According to [BoyerLindquist]*31 the Kerr metric is described by the authors R. H. Boyer and R. W. Lindquist as

\[
ds^2 = dr^2 + 2a \sin^2 \vartheta dr d\phi + (r^2 + a^2) \sin^2 \vartheta d\phi^2 + \\
+ \Sigma d\vartheta^2 - dt^2 + (2Mr/\Sigma)(dr + a \sin^2 \vartheta d\phi + dt)^2, \quad [BoyerLindquist](2.7)
\]

with

\[
\Sigma = r^2 + a^2 \cos^2 \vartheta. \quad [BoyerLindquist](2.8)
\]

But \( r \) is a function of \( x, y, z \) of the standard co-ordinate system \( (t, x, y, z) \) such that

\[
\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \quad [BoyerLindquist](2.5)
\]

---

Therefore the surface \( r = \text{constant} \) referred in the above [Collas] Footnote (5) are confocal ellipsoids. ([BoyerLindquist] p.269L, the last line.) In other words, [Collas] says that **ellipsoids are suitable figures for \( S \) to solve the problem (Q2).**

However it seems doubtful that an exact ellipsoid is suitable for \( S \), since it seems doubtful that the vacuum boundary \( \{ r = r_+(\zeta) \} \) of the interior solution we have constructed would be an exact ellipsoid if \( \Omega \neq 0 \). Let us explain the reason of this doubt.
We have

\[ r_+(\zeta) = a \left[ \Xi_1\left(\zeta, \frac{1}{\gamma - 1}, b\right) + O\left(\frac{u_0^2}{c^2}\right) \right], \tag{3.118} \]

where

\[ a = \sqrt{\frac{A \gamma}{4\pi G(\gamma - 1)} \rho_0^{\frac{2-\gamma}{2}}} , \quad b = \frac{\Omega^2}{4\pi G \rho_0} \]

and \( \xi = \Xi_1\left(\zeta, \frac{1}{\gamma - 1}, b\right) \) is the surface curve of the distorted Lane-Emden function \( \Theta\left(\xi, \zeta, \frac{1}{\gamma - 1}, b\right) \), while

\[ u_N(r, \zeta) = u_0 \Theta\left(\frac{r}{a}, \zeta, \frac{1}{\gamma - 1}, b\right). \]

On the distorted Lane-Emden function, as shown in [?, §7] and [?, §5],
we have the approximation

\[ \Xi_1(\zeta) = \xi_1 + \frac{\xi_1^2}{\mu_1} h(\xi_1, \zeta) b + O(b^\frac{1}{\gamma-1} \zeta^2), \]

where \( \xi_1 = \xi_1 \left( \frac{1}{\gamma-1} \right), \mu_1 = \mu_1 \left( \frac{1}{\gamma-1} \right) \) are positive numbers,

\[ h(\xi, \zeta) = h_0(\xi) + A_2 \psi_2(\xi) P_2(\zeta), \]

\( A_2 < 0, \psi_2(\xi_1) > 0, \) and

\[ P_2(\zeta) = \frac{1}{2} (3\zeta^2 - 1). \]

Thus \( r = r_+(\zeta) \) is approximated by

\[ r = a \left[ c_0 + (c_1 - c_2 \zeta^2) b \right] \]
with $c_2 > 0$, which does not give an ellipsoid if $b \neq 0$, since an ellipsoid should be given by a function of the form

$$r = \frac{a_0}{\sqrt{1 + a_1 \zeta^2}}.$$

Of course this is not a rigorous proof of the claim that $r = r_+(\zeta)$ does not give an ellipsoid, since the detailed structure of the remainder terms $O(u_0^2/c^2)$ and $O(b \frac{1}{\gamma-1} \zeta^2)$ are not clearly analyzed. But it is strongly plausible that $\xi = \Xi_1(\zeta)$ does not give an ellipsoid if $b \neq 0$. 
Actually the following “no-go theorem” has been known for more than 120 years:

**Theorem of Hamy-Pizzetti:** An ellipsoidal stratification (that is, the situation that all level surfaces \( \{ u = \text{Const.} \} \) are ellipsoids) is impossible for heterogeneous (that is, with non-constant \( \rho \)), rotating symmetric figures of equilibrium (that is, axisymmetric stationary solutions of the Euler-Poisson equations with \( \Omega \neq 0 \)).

See Sec. 3.2 of [Moritz]*32, [Pizzetti]*33, [Wavre]*34.

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*32 H. Moritz, *The Figure of the Earth*, Wichmann, Karlsruhe, 1990.
By this no-go theorem it is impossible that all level surfaces \( \{ \rho = \text{Const.} \} \) are ellipsoids, since \( \partial \Theta / \partial \xi < 0 \) for \( 0 < \xi < \Xi_1(\zeta) \) so that \( \partial \rho / \partial r < 0 \) for \( 0 < r < r_+(\zeta) \) in the non-relativistic problem for which \( c = \infty, \ r_+(\zeta) = a \Xi_1(\zeta) \), and

\[
\rho = \rho_N = \left( \frac{\gamma - 1}{A \gamma} \right)^{\frac{1}{\gamma - 1}} \ (u_N \vee 0)^{\frac{1}{\gamma - 1}} = \rho_0(\Theta \vee 0)^{\frac{1}{\gamma - 1}},
\]

although this no-go theorem does not claim that it is impossible that the individual level surface \( \{ u(= u_N) = 0 \} = \{ r = r_+(\zeta)(= a \Xi_1(\zeta)) \} \) is an ellipsoid.
3.7.5 Solution by J. L. Hernandez-Pastora and L Herrera

On the last Tuesday, July 23, 2019, Professor Luis Herrera kindly drew my attention to his recent paper [HernandezPastora-Herrera2017]*35. There are constructed exact solutions to the matter-vacuum matching problem with the Kerr metric.

These metric is of the form in the interior \( \{ r < r_\Sigma \} \):

\[
ds^2 = e^{2F} (dt + A d\phi)^2 - e^{-2\hat{F} + 2\hat{K}} dr^2 \\
- e^{2\hat{F} + 2\hat{K}} r^2 d\theta^2 - e^{-2\hat{F}} r^2 \sin^2 \theta d\phi^2,
\]

and, in the exterior \( \{ r > r_\Sigma \} \):

\[
ds^2 = e^{2F} Z^2 (dt + A d\phi)^2 - e^{-2\hat{F} + 2\hat{K}} W^{-1} dr^2 \\
- e^{2\hat{F} + 2\hat{K}} r^2 d\theta^2 - e^{-2\hat{F}} r^2 \sin^2 \theta d\phi^2,
\]

which satisfies the \( C^1 \)-matching conditions across the surface \( \{ r = r_\Sigma \} \).
Here \((r, \theta)\) is the Erez-Rosen coordinates:

\[
\varpi = \sqrt{r(r-M)} \sin \theta, \quad z = (r-M) \cos \theta,
\]

and \(\hat{F} = F - F^S, \hat{K} = K - K^S\) with the Schwarzschild metric

\[
e^{2F^S} = 1 - \frac{2M}{r},
\]

\[
e^{2K^S} = \left(1 - \frac{2M}{r}\right) \left[\left(1 - \frac{M}{r}\right)^2 - \frac{M^2}{r^2} \cos \theta\right]^{-1},
\]

while \(r_\Sigma > 2M\),

\[
W = W(r) = 1 - pr^2. \quad p = \frac{2M}{r_\Sigma^3} \text{ is a positive constant},
\]

\[
Z = Z(r) = \frac{1}{2} (3\sqrt{W(r)} - \sqrt{W(r)}),
\]

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and the exterior metric is nothing but the Kerr metric, say,

\[
e^{2F} = \frac{(r_+ + r_-)^2(1 - j^2) - 4M^2(1 - j^2) + j^2(r_+ - r_-)^2}{(r_+ + r_- + 2M)^2(1 - j^2) + j^2(r_+ - r_-)^2},
\]

\[
e^{2K} = \frac{(r_+ + r_-)^2(1 - j^2) - 4M^2(1 - j^2) + j^2(r_+ - r_-)^2}{4r_+r_- (1 - j^2)},
\]

\[
A = \frac{j(2M + r_+ + r_-)((r_+ - r_-)^2 - 4M^2(1 - j^2))}{(r_+ + r_-)^2(1 - j^2) - 4M^2(1 - j^2) + j^2(r_+ - r_-)^2},
\]

where \( j = J/M^2 = a/M \), \( a \) being the parameter in the Boyer-Lindquist coordinates expression,

\[
r^2_\pm = \omega^2 + (z \pm M \sqrt{1 - j^2})^2 \\
= [r - M \pm M \sqrt{1 - j^2 \cos \theta}]^2 - M^2 j^2 \sin^2 \theta.
\]
However the energy-momentum tensor of the interior metric is

\[
\begin{align*}
T^t_t &= -\kappa (8\pi \rho + p_{zz} - E + 3\beta J_+ + \beta AI), \\
T^r_r &= \kappa (8\pi P - p_{xx} - \beta J_-), \\
T^\theta_\theta &= \kappa (8\pi P + p_{xx} + \beta J_-), \\
T^\phi_\phi &= \kappa (8\pi P - p_{zz} + \beta J_+ + \beta AI), \\
T^\phi_t &= -\kappa \beta I, \\
T^\theta_r &= -\frac{\kappa}{r^2} \left[ \cdots \right],
\end{align*}
\]

with

\[
\begin{align*}
\rho &= \frac{3p}{8\pi}, & P &= \rho \frac{\sqrt{W(r)} - \sqrt{W(r_\Sigma)}}{3\sqrt{W(r_\Sigma)} - \sqrt{W(r)}}, \\
\kappa &= \frac{1}{8\pi} e^{\hat{F} - 2\hat{K}}, & \beta &= e^{4\hat{F}} Z^2,
\end{align*}
\]
etc. The detailed are omitted, but we note \( I = J_{\pm} = 0 \) when \( A = 0 \), and
\[
E = -2\Delta F + (1 - W)[\cdots].
\]

It is supposed that \( r_{\Sigma}/M > 9/4 \) in order that \( P > 0 \), or \( r_{\Sigma}/M > 8/3 \) in order that \( P > \rho \).

So, according to a private communication with Professor Luis Herrera, the pressure is anisotropic, that is, either \( p_{xx} \) or \( p_{zz} \) does not vanish for the concrete solution constructed in [HernandezPastoraHerrera2017].

Of course our Assumption 1 on the equation of states requires isotropic pressure.
THANK YOU

FOR YOUR ATTENTION!

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Please visit my Homepage:

“Arkivo de Tetu Makino”
(http://hc3.seikyou.ne.jp/home/Tetu.Makino)