On mathematical study of the Einstein-Euler-de Sitter equations

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1 Introduction

The Einstein-de Sitter equation:

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}R_{\alpha\beta}) - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \]

The energy-momentum tensor of a perfect fluid

\[ T^{\mu\nu} = (c^2 \rho + P)U^\mu U^\nu - Pg^{\mu\nu}. \]
**Assumption.** $P$ is a given analytic function of $\rho > 0$ such that $0 < P, 0 < dP/d\rho < c^2$ for $\rho > 0$, and $P \to 0$ as $\rho \to +0$. Moreover there are positive constants $A, \gamma$ and an analytic function $\Omega$ on a neighborhood of $[0, +\infty[$ such that $\Omega(0) = 1$ and

$$P = A\rho^\gamma \Omega(A\rho^{\gamma-1}/c^2).$$

We assume that $1 < \gamma < 2$ and $\frac{1}{\gamma - 1}$ is an integer.
Spherically symmetric metric:

\[ ds^2 = e^{2F(t, r)} c^2 dt^2 - e^{2H(t, r)} dr^2 - R(t, r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

co-moving:

\[ U^t = e^{-F}, \quad U^r = U^\theta = U^\phi = 0. \]
Equations on \( \{ \rho > 0 \} \):

\[
e^{-F} \frac{\partial R}{\partial t} = V \\
\ \\
e^{-F} \frac{\partial V}{\partial t} = -GR \left( \frac{m}{R^3} + \frac{4\pi P}{c^2} \right) + \frac{c^2 \Lambda}{3} R + \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right) \frac{P'}{R' (\rho + P/c^2)} \tag{1b}
\]

\[
e^{-F} \frac{\partial \rho}{\partial t} = -(\rho + P/c^2) \left( \frac{V'}{R'} + \frac{2V}{R} \right) \tag{1c}
\]

\[
e^{-F} \frac{\partial m}{\partial t} = -\frac{4\pi}{c^2} R^2 PV \tag{1d}
\]

Here \( X' \) stands for \( \partial X/\partial r \).
Put

\[ m = 4\pi \int_0^r \rho R^2 R' dr, \]

supposing that \( \rho \) is continuous at \( r = 0 \).

\[ e^{2H} = \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} \frac{R^2}{R'} \right)^{-1} (R')^2. \]

\[ e^{2F} = \kappa e^{-2u/c^2} \]

with

\[ u := \int_0^\rho \frac{dP}{\rho + P/c^2} = \frac{\gamma A}{\gamma - 1} \rho^{\gamma-1} \Omega_u(A\rho^{\gamma-1}/c^2). \]
Note

\[ \rho = A_1 u^{\frac{1}{\gamma-1}} \Omega_\rho (u/c^2), \quad P = AA_1^\gamma u^{\frac{\gamma}{\gamma-1}} \Omega_P (u/c^2) \]

with \( A_1 := \left( \frac{\gamma - 1}{\gamma A} \right)^{\frac{1}{\gamma-1}} \).
2 Equilibrium

The Tolman-Oppenheimer-Volkoff-de Sitter equation:

\[
\frac{dm}{dr} = 4\pi r^2 \rho, \quad (2a)
\]

\[
\frac{dP}{dr} = -\left(\rho + \frac{P}{c^2}\right)\frac{G\left(m + \frac{4\pi r^3}{c^2}P\right) - \frac{c^2 \Lambda}{3} r^3}{r^2 \left(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2\right)}. \quad (2b)
\]
For arbitrary positive central density $\rho_c$ there exists a unique solution germ $(m(r), P(r))$, $0 < r \ll 1$, such that

\begin{align}
m &= \frac{4\pi}{3} \rho_c r^3 + [r^2]_2 r, \\
P &= P_c - (\rho_c + P_c/c^2) \left( \frac{4\pi G}{3} (\rho_c + 3P_c/c^2) - \frac{c^2 \Lambda}{3} \right) \frac{r^2}{2} + [r^2]_2. \tag{3b}
\end{align}

Here $[X]_Q$ denotes a convergent power series of the form $\sum_{k \geq Q} a_k X^k$.

We denote

\begin{align}
\kappa(r,m) &:= 1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2, \\
Q(r, m, P) &:= G \left( m + \frac{4\pi r^3}{c^2} P \right) - \frac{c^2 \Lambda}{3} r^3.
\end{align}
Definition 1. A solution \((m(r), P(r)), 0 < r < r_+, \) such that \(\rho > 0, \kappa(r, m) > 0\) of (2a)(2b) is said to be \textbf{monotone-short} if \(r_+ < \infty,\) 
\(dP/dr < 0\) for \(0 < r < r_+,\) that is, \(Q(r, m(r), P(r)) > 0,\) and \(P \to 0\)
as \(r \to r_+ - 0\) and if

\[
\kappa_+ := \lim_{r \to r_+ - 0} \kappa(r, m(r)) = 1 - \frac{2Gm_+}{c^2 r_+} - \frac{\Lambda}{3} r_+^2
\]

and

\[
Q_+ := \lim_{r \to r_+ - 0} Q(r, m(r), P(r)) = Gm_+ - \frac{c^2 \Lambda}{3} r_+^3
\]

are positive. Here

\[
m_+ := \lim_{r \to r_+ - 0} m(r) = 4\pi \int_0^{r_+} \rho(r) r^2 dr.
\]
Suppose that there is a monotone-short solution \((\tilde{m}(r), \tilde{P}(r)), 0 < r < r_+,\) satisfying (3a)(3b), and fix it. Then the associated function \(u = \bar{u}(r)\) turns out to be analytic on a neighborhood of \([0, r_+]\) and

\[
\bar{u}(r) = \frac{Q_+}{r_+^2 \kappa_+} (r_+ - r) + [r_+ - r]_2
\]

as \(r \to r_+ - 0.\)
3 Equations for the small perturbation from the equilibrium

Solutions of (1a)-(1d) of the form

\[ R = r(1 + y), \quad V = rv \]

with small unknowns \( y, v \)
\[ e^{-F} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right)v, \tag{4a} \]

\[ e^{-F} \frac{\partial v}{\partial t} = \frac{(1 + y)^2}{c^2} \frac{P}{\bar{\rho}} v \frac{\partial}{\partial r} (rv) + \]

\[ -G(1 + y) \left( \frac{m}{r^3 (1 + y)^3} + \frac{4\pi}{c^2 P} \right) + \frac{c^2 \Lambda}{3} (1 + y) + \]

\[ -\left(1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r (1 + y)} - \frac{\Lambda}{3} r^2 (1 + y)^2 \right) \times \]

\[ \times \left(1 + \frac{P}{c^2 \rho}\right)^{-1} (1 + y)^2 \frac{\partial P}{\bar{\rho} r} \frac{\partial}{\partial r}. \tag{4b} \]

Here \( m = \bar{m}(r) \) is a given function and \( \rho \) is considered as given functions of \( r \) and the unknowns \( y, z := r \partial y / \partial r \) as

\[ \rho = \bar{\rho}(r)(1 + y)^{-2}(1 + y + z)^{-1} \]
4 Analysis of the linearized equation

The linearized wave equation of (4a)(4b):

\[ \frac{\partial^2 y}{\partial t^2} + \mathcal{L}y = 0 \quad \text{with} \quad \mathcal{L}y = -\frac{1}{b} \frac{d}{dr} a \frac{dy}{dr} + Qy, \]

where

\[ a = e^{\tilde{H} + \tilde{F}} \frac{\Gamma r^4}{1 + \rho} \frac{P}{c^2 \rho}, \]

\[ b = e^{3 \tilde{H} - \tilde{F}} \frac{r^4 \rho}{1 + \rho}, \]

\[ Q \in C([0, r_+]). \]
Proposition 1. The operator $\mathcal{T}_0, \mathcal{D}(\mathcal{T}_0) = C_0^\infty(0, r_+), \mathcal{T}_0y = Ly$ in the Hilbert space $L^2((0, r_+); b(r)dr)$ admits the Friedrichs extension $\mathcal{T}$, a self adjoint operator, whose spectrum consists of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_\nu < \cdots \to +\infty$. 
\[ x := \frac{\tan^2 \theta}{1 + \tan^2 \theta} \quad \text{with} \quad \theta := \frac{\pi}{2\xi_+} \int_0^r \sqrt{\frac{\bar{\rho}}{\Gamma P}} e^{-\bar{\Phi} + \bar{H}} \, dr. \]

\[ r = C_0 \sqrt{x}(1 + [x]_1) \quad \text{as} \quad x \to 0, \]
\[ r_+ - r = C_1 (1 - x)(1 + [1 - x]_1) \quad \text{as} \quad x \to 1 \]

\[ \mathcal{L}y = -x(1-x) \frac{d^2 y}{dx^2} - \left( \frac{5}{2} (1-x) - \frac{N}{2} x \right) \frac{dy}{dx} + L_1(x) x(1-x) \frac{dy}{dx} + L_0(x) y, \]

Here \( L_1(x), L_0(x) \) are analytic functions on a neighborhood of \([0, 1]\), and

\[ N := \frac{2\gamma}{\gamma - 1}. \]
Note

\[
X \frac{d^2}{dX^2} + \frac{N}{2} \frac{d}{dX} = \frac{d^2\xi}{d\xi^2} + \frac{N - 1}{\xi} \frac{d}{d\xi} = \Delta^{(N)}_{\xi} \quad \text{for} \quad X = \frac{\xi^2}{4}
\]
5 Rewriting (4a)(4b) using $\mathcal{L}$

Putting

$$J := e^F (1 + P/c^2 \rho),$$

we rewrite the system of equations (4a)(4b) as

\[
\frac{\partial y}{\partial t} - Jv = 0, \quad \text{(5a)}
\]

\[
\frac{\partial v}{\partial t} + H_1 \mathcal{L}y + H_2 = 0. \quad \text{(5b)}
\]
Here $H_1, H_2$ are analytic functions of $x$ in a neighborhood of $[0, 1]$ and $y, z = x\partial y/\partial x, v, w = x\partial v/\partial x$ in a neighborhood of $(0, 0, 0, 0)$. Moreover

$$H_1(x, 0, 0, 0) = \frac{1}{J(x, 0, 0, 0)}$$

and

$$H_2(x, 0, \cdots, 0) = \partial_y H_2(x, 0, \cdots, 0) = \cdots = \partial_w H_2(x, 0, \cdots, 0) = 0.$$
6 Main results

(I). Let us fix a time periodic solution of the linearized equation:

\[ Y_1 = \sin(\sqrt{\lambda}t + \Theta_0)\psi(x), \]

where \( \lambda \) is a positive eigenvalue of the operator \( \mathcal{T} \) and \( \psi \) is an associated eigenfunction. We seek a solution of the form

\[ R = r(1+y) = r(1 + \varepsilon Y_1 + \varepsilon^2 \dot{y}), \quad V = r(\varepsilon V_1 + \varepsilon^2 \dot{v}), \]

where

\[ V_1 = e^{-\bar{F}}(1 + \frac{P}{c^2 \rho})^{-1} \frac{\partial Y_1}{\partial t}. \]
Theorem 1. Given $T > 0$, there is a positive number $\epsilon_0$ such that, for $|\epsilon| \leq \epsilon_0$, there is a solution $(\tilde{y}, \tilde{v}) \in C^\infty([0, T] \times [0, 1])$ such that

$$
\sup_{j+k \leq n} \left\| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^k (\tilde{y}, \tilde{v}) \right\|_{L^\infty([0,T] \times [0,1])} \leq C(n).
$$
Note that

\[ R(t, r_+) = r_+ (1 + \varepsilon \sin(\sqrt{\lambda t} + \Theta_0) + O(\varepsilon^2)), \]

provided that \( \psi \) has been normalized as \( \psi(x = 1) = 1 \), and that the density distribution enjoys the ‘physical vacuum boundary’ condition:

\[
\rho(t, r) = \begin{cases} 
C(t)(r_+ - r) \frac{1}{r_+^{\gamma-1}} (1 + O(r_+ - r)) & (0 \leq r < r_+) \\
0 & (r_+ \leq r)
\end{cases}
\]

with a smooth function \( C(t) \) of \( t \) such that

\[
C(t) = \left( \frac{\gamma - 1}{A\gamma} \frac{Q_+}{r_+^2 \kappa_+} \right) \frac{1}{r_+^{\gamma-1}} + O(\varepsilon).
\]
Also we can consider the Cauchy problem

\[
\frac{\partial y}{\partial t} - Jv = 0, \quad \frac{\partial v}{\partial t} + H_1 \mathcal{L}y + H_2 = 0, \\
y\big|_{t=0} = \psi_0(x), \quad v\big|_{t=0} = \psi_1(x).
\]

Then we have

**Theorem 2.** Given \( T > 0 \), there exits a small positive \( \delta \) such that if \( \psi_0, \psi_1 \in C^\infty([0, 1]) \) satisfy

\[
\max_{k \leq \mathfrak{K}} \left\{ \left\| \left( \frac{d}{dx} \right)^k \psi_0 \right\|_{L^\infty}, \left\| \left( \frac{d}{dx} \right)^k \psi_1 \right\|_{L^\infty} \right\} \leq \delta,
\]

then there exists a unique solution \((y, v)\) of the Cauchy problem in \( C^\infty([0, T] \times [0, 1]) \). Here \( \mathfrak{K} \) is sufficiently large number.
7 Metric in the exterior domain

The Schwarzschild-de Sitter metric:

\[
    ds^2 = \kappa^\# c^2 (dt^\#)^2 - \frac{1}{\kappa^\#} (dR^\#)^2 - (R^\#)^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

Here \( t^\# = t^\#(t, r) \), \( R^\# = R^\#(t, r) \) are smooth functions of \( 0 \leq t \leq T, r_+ \leq r \leq r_+ + \delta \), \( \delta \) being a small positive number, and

\[
    \kappa^\# = 1 - \frac{2Gm_+}{c^2 R^\#} - \frac{\Lambda}{3} (R^\#)^2.
\]

The patched metric:

\[
    ds^2 = g_{00}c^2 dt^2 + 2g_{01}c dt dr + g_{11} dr^2 + g_{22} (d\theta^2 + \sin^2 \theta d\phi^2),
\]
where

\[
\begin{align*}
g_{00} &= \begin{cases} 
  e^{2F} = \kappa_+ e^{-2u/c^2} & (r \leq r_+) \\
  \kappa_\# (\partial_t t\#)^2 - \frac{1}{c^2 \kappa_\#} (\partial_t R\#)^2 & (r_+ < r)
\end{cases} \\
g_{01} &= \begin{cases} 
  0 & (r \leq r_+) \\
  c\kappa_\# (\partial_t t\#)(\partial_r t\#) - \frac{1}{c\kappa_\#} (\partial_t R\#)(\partial_r R\#) & (r_+ < r)
\end{cases} \\
g_{11} &= \begin{cases} 
  -e^{2H} = - \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-1} (\partial_r R)^2 & (r \leq r_+) \\
  c^2 \kappa_\# (\partial_r t\#)^2 - \frac{1}{\kappa_\#} (\partial_r R\#)^2 & (r_+ < r)
\end{cases} \\
g_{22} &= \begin{cases} 
  -R^2 & (r \leq r_+) \\
  -(R\#)^2 & (r_+ < r)
\end{cases}
\end{align*}
\]
Let $R = R^\#$ and $\partial_r R = \partial_r R^\#$ along $r = r_+$ in order that $g_{22}$ be of class $C^1$.

Moreover

$$\frac{\partial t^\#}{\partial t}, \quad \frac{\partial t^\#}{\partial r}, \quad \frac{\partial^2 t^\#}{\partial r^2}, \quad \frac{\partial^2 R^\#}{\partial r^2} \quad \text{at} \quad r = r_+ + 0$$

are uniquely determined in order that $g_{00}, g_{01}, g_{11}$ be of class $C^1$ across $r = r_+$. 


\[
\frac{\partial^2 R^\#}{\partial r^2} \bigg|_{r_+0} - \frac{\partial^2 R}{\partial r^2} \bigg|_{r_+-0} = A \left( \frac{\partial R}{\partial r} \right)^2,
\]

with

\[
A = -\frac{V^2}{c^2} \left( \left( \frac{Gm_+}{c^2 R^2} - \frac{\Lambda}{3} R + \frac{1}{\sqrt{\kappa_+}} \frac{1}{c^2} \frac{\partial V}{\partial t} \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} - \frac{\Lambda}{3} R^2 \right) \right)^{-2} \bigg|_{r_+-0}.
\]

\[
\frac{\partial^2 R^\#}{\partial r^2} \equiv \frac{\partial^2 R}{\partial r^2} \iff \frac{\partial R}{\partial t} \equiv 0 \quad \text{at} \quad r = r_+
\]
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