Mathematical Theory of Rotating Gaseous Stars

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Einstein-Euler equations

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1) \]

\[ T^{\mu\nu} = (c^2 \rho + P) U^\mu U^\nu - Pg^{\mu\nu} \quad (2) \]

for the metric \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \)
Assumption (A):

$P$ is a given function of $\rho > 0$ such that $0 < P, 0 < dP/d\rho < c^2$ for $\rho > 0$;

there exists a smooth function $\Lambda$ which is analytic near 0, $\Lambda(0) = 0$, and

\[ P = A\rho^{\gamma}(1 + \Lambda(A\rho^{\gamma^{-1}}/c^2)) \]  \hspace{1cm} (3)

Here $A, \gamma$ are positive constants, and

\[ \frac{6}{5} < \gamma < \frac{3}{2}. \]
Result of [1]: Newtonian problem governed by the Euler-Poisson equations: \( c = +\infty \) admits axially and equatorially symmetric slowly rotating solutions with the density distribution

\[
\rho_N(r, \zeta) = \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma-1}} \max(u_N(r, \zeta), 0)^{\frac{1}{\gamma-1}}
\]

with compact support, where \( r = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} \), \( \zeta = x_3/r \) for \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), and the velocity field

\[
\vec{v}_N = -\Omega x_2 \frac{\partial}{\partial x_1} + \Omega x_1 \frac{\partial}{\partial x_2}
\]

with sufficiently small constant angular velocity \( \Omega \).
**Problem:** Find a solution of the Einstein-Euler equations, which tends to the solution $\rho_N, \vec{v}_N$ as $c \to \infty$ of the form:

$$ds^2 = e^{2F}(cdt + A\phi)^2 - e^{-2F}[e^{2K}(d\varpi^2 + dz^2) + \Pi^2 d\phi^2]$$  \hspace{1cm} (4)

Here $x_1 = \varpi \cos \phi, x_2 = \varpi \sin \phi, x_3 = z$. 
Main Result:

When $u_O/c^2$ is sufficiently small, we have a solution

$$F = \frac{1}{c^2} \left( \Phi_N - \frac{\Omega^2}{2} \varpi^2 \right) + O(1/c^4), \quad K = -\frac{\Omega^2}{2c^2} \varpi^2 + O(1/c^4),$$

$$A = -\varpi^2 \frac{\Omega}{c} (1 + O(1/c^2)), \quad \Pi = \varpi (1 + O(1/c^4)),$$

$$\rho = \left( \frac{\gamma - 1}{A\gamma} \right)^\frac{1}{\gamma - 1} \max(u, 0) \frac{1}{\gamma - 1} (1 + O(1/c^2)), \quad u = u_N + O(1/c^2) \quad (5)$$

Here $\Phi_N$ is the Newton potential generated by $\rho_N$, and $u_O = u_N(0, 0)$. 


The equations to be solved reduce to

\[
\frac{\partial^2 F}{\partial \varpi^2} + \frac{\partial^2 F}{\partial z^2} + \frac{1}{\Pi} \left( \frac{\partial F}{\partial \varpi} \frac{\partial \Pi}{\partial \varpi} + \frac{\partial F}{\partial z} \frac{\partial \Pi}{\partial z} \right) + \frac{e^{4F}}{2\Pi^2} \left[ \left( \frac{\partial A}{\partial \varpi} \right)^2 + \left( \frac{\partial A}{\partial z} \right)^2 \right] \\
= \frac{4\pi G}{c^4} e^{-2F+2K}(\epsilon + 3P), \tag{6a}
\]

\[
\frac{\partial}{\partial \varpi} \left( \frac{e^{4F}}{\Pi} \frac{\partial A}{\partial \varpi} \right) + \frac{\partial}{\partial z} \left( \frac{e^{4F}}{\Pi} \frac{\partial A}{\partial z} \right) = 0, \tag{6b}
\]

\[
\frac{\partial^2 \Pi}{\partial \varpi^2} + \frac{\partial^2 \Pi}{\partial z^2} = \frac{16\pi G}{c^4} e^{-2F+2K} \Pi, \tag{6c}
\]

\[
\frac{\partial \Pi}{\partial \varpi} \frac{\partial K}{\partial \varpi} - \frac{\partial \Pi}{\partial z} \frac{\partial K}{\partial z} = \frac{1}{2} \left( \frac{\partial^2 \Pi}{\partial \varpi^2} - \frac{\partial^2 \Pi}{\partial z^2} \right) + \Pi \left[ \left( \frac{\partial F}{\partial \varpi} \right)^2 - \left( \frac{\partial F}{\partial z} \right)^2 \right] + \frac{e^{4F}}{4\Pi} \left[ \left( \frac{\partial A}{\partial \varpi} \right)^2 - \left( \frac{\partial A}{\partial z} \right)^2 \right], \tag{6d}
\]

\[
\frac{\partial \Pi}{\partial z} \frac{\partial K}{\partial \varpi} + \frac{\partial \Pi}{\partial \varpi} \frac{\partial K}{\partial z} = \frac{\partial^2 \Pi}{\partial \varpi \partial z} + 2\Pi \frac{\partial F}{\partial \varpi} \frac{\partial F}{\partial z} - \frac{e^{4F}}{2\Pi} \frac{\partial A}{\partial \varpi} \frac{\partial A}{\partial z}, \tag{6e}
\]

\[
F = -\frac{u}{c^2} + \text{Const..} \tag{6f}
\]
(6a),(6b),(6c) are elliptic equations on $F, A, \Pi$ when $K$ is given, and (6d),(6e) are a first order system on $K$ when $F, A, \Pi$ are given.

**[Point 1]:** When $P = 0$, the integrability condition of (6d)(6e) is guaranteed a priori, but when $P \neq 0$, it is not the case and a device is needed.

Anyway we apply the fixed point theorem for contraction mappings by setting appropriate functional spaces.

**[Point 2]:** Through this process, we need the crucial lemma of [1] in order to prove the solvability of the elliptic equation on $F$. 
[Point 1]:

\[
(6d),(6e) \quad \Leftrightarrow \quad \frac{\partial K}{\partial \varpi} = \tilde{K}_1, \quad \frac{\partial K}{\partial z} = \tilde{K}_3,
\]

where

\[
\tilde{K}_1 = \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \left( \frac{\partial \Pi}{\partial \varpi} \cdot \text{RH}(6d) + \frac{\partial \Pi}{\partial z} \cdot \text{RH}(6e) \right), \quad (7a)
\]

\[
\tilde{K}_3 = \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \left( -\frac{\partial \Pi}{\partial z} \cdot \text{RH}(6d) + \frac{\partial \Pi}{\partial \varpi} \cdot \text{RH}(6e) \right). \quad (7b)
\]

But

\[
\frac{\partial \tilde{K}_1}{\partial z} - \frac{\partial \tilde{K}_3}{\partial \varpi} = \frac{8\pi G}{c^4} e^{-2F+2K} P_{\Pi} \left[ \left( \frac{\partial \Pi}{\partial \varpi} \right)^2 + \left( \frac{\partial \Pi}{\partial z} \right)^2 \right]^{-1} \times
\]

\[
\times \left[ \left( \frac{\partial K}{\partial \varpi} - \tilde{K}_1 \right) \frac{\partial \Pi}{\partial z} - \left( \frac{\partial K}{\partial z} - \tilde{K}_3 \right) \frac{\partial \Pi}{\partial \varpi} \right]. \quad (8)
\]
Lemma 1: Even if $P \neq 0$, we have

$$\tilde{K} = K \quad \Rightarrow \quad \frac{\partial K}{\partial \varpi} = \tilde{K}_1, \quad \frac{\partial K}{\partial z} = \tilde{K}_3 \quad \Rightarrow \quad (6d), (6e),$$

where

$$\tilde{K}(\varpi, z) := \int_0^z \tilde{K}_3(0, z') dz' + \int_0^\varpi \tilde{K}_1(\varpi', z) d\varpi' \quad (9)$$
[Point 2]

Post-Newtonian approximation:

\[ F = \frac{1}{c^2} \left( \Phi_N - \frac{\Omega^2}{2} \right) - \frac{w}{c^4}, \] (10a)

\[ A = \left( -\frac{\Omega}{c} + \frac{Y}{c^3} \right) \varpi^2, \] (10b)

\[ \Pi = \varpi \left( 1 + \frac{X}{c^4} \right), \] (10c)

\[ K = -\frac{\Omega^2}{2c^2} \varpi^2 + \frac{V}{c^4} \] (10d)

\[ u = u_N + \frac{w}{c^2} \] (10e)

\[ \Rightarrow \quad \rho = \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma - 1}} \max(u, 0)^{\frac{1}{\gamma - 1}} (1 + [u/c^2]_1) \]
\[
\left[ \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial z^2} + \frac{1}{(\gamma - 1) a^2} \max \left( \frac{u_N}{u_O}, 0 \right)^{\frac{1}{\gamma - 1} - 1} \right] w = \\
= 8 \left( \Phi_N - \frac{\Omega^2}{2} \omega^2 \right) \Omega^2 - 2 \Omega \left( 2Y + \omega \frac{\partial Y}{\partial \omega} \right) + \\
- 4\pi G \left( -2\Phi_N \rho_N + \left( \frac{\gamma}{\gamma - 1} \Lambda_1 + 3 \right) P_N \right) + R_a, \quad (11a) \\
\left[ \frac{\partial^2}{\partial \omega^2} + \frac{3}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial z^2} \right] Y = \frac{8}{\omega} \frac{\partial}{\partial \omega} \left[ \Phi_N - \frac{\Omega^2}{2} \omega^2 \right] \Omega + R_b, \quad (11b) \\
\left[ \frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial z^2} \right] X = 16\pi G P_N + R_c, \quad (11c)
\]
Lemma 2: Given axially and equatorially symmetric, compactly supported function \( g \), the integral equation

\[
Q = \mathcal{K}\left[\frac{1}{(\gamma - 1)} \max \left( \frac{u_N}{u_O}, 0 \right)^{\frac{1}{\gamma - 1}} Q + g \right]
\]

admits a unique axially and equatorially symmetric solution \( Q \). Here

\[
\mathcal{K} f(x) = \frac{1}{4\pi} \int \frac{f(x')}{|x - x'|} dx' - \frac{1}{4\pi} \int \frac{f(x')}{|x'|} dx'.
\]

Note that then

\[
\left[ \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} + \frac{1}{(\gamma - 1)} \max \left( \frac{u_N}{u_O}, 0 \right)^{\frac{1}{\gamma - 1}} \right] Q + g = 0,
\]

\( Q(0, 0) = 0 \)
Open problem:

The solution is constructed on a bounded domain which contains the support of $\rho$. The matching problem to the exterior vacuum metric which is defined on the whole space and asymptotically flat at the space infinity.

The preprint is available at arXiv [2].

参考文献

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FOR YOUR ATTENTION!

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