球対称気体星の非動径振動について

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1 Problem

Euler-Poisson equations on \((t, x) = (t, x_1, x_2, x_3) \in [0, +\infty[ \times \mathbb{R}^3:\)

\[
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x^k} (\rho v^k) = 0, \tag{1a}
\]

\[
\rho \left( \frac{\partial v^j}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial v^j}{\partial x^k} \right) + \frac{\partial P}{\partial x^j} + \rho \frac{\partial \Phi}{\partial x^j} = 0, \quad (j = 1, 2, 3), \tag{1b}
\]

\[
\rho \left( \frac{\partial S}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial S}{\partial x^k} \right) = 0, \tag{1c}
\]

\[
\triangle \Phi = 4\pi G \rho. \tag{1d}
\]
Supposing that the support of $\rho(t, \cdot)$ is compact, we replace the Poisson equation (1d) by the Newtonian potential

$$\Phi = -4\pi G \mathcal{K} \rho(t, \cdot),$$  \hspace{1cm} (2)

where the integral operator $\mathcal{K}$ is defined as

$$\mathcal{K} f(x) = \frac{1}{4\pi} \int \frac{f(x')}{\|x - x'\|} dx'.$$ \hspace{1cm} (3)
Assumption 1.

$$P = \rho^\gamma \exp \left( \frac{S}{C_V} \right)$$  \hfill (4)

for $\rho > 0$, where $\gamma, C_V$ are positive constants such that $1 < \gamma < 2$. 
Definition 1. admissible spherically symmetric equilibrium 

\((\tilde{\rho}, \tilde{S})\) a function of \(r = \|\mathbf{x}\| : \tilde{\rho} \in C^1_0(\mathbb{R}^3, [0, +\infty]), \quad \tilde{S} \in C^1(\mathbb{R}^3, \mathbb{R}), \)

\(\rho = \tilde{\rho}, S = \tilde{S}, \mathbf{v} = 0\) satisfies (1)(2), \(\exists R < \infty\) such that

1) \(\{\tilde{\rho} > 0\} = B_R(:= \{\mathbf{x} \in \mathbb{R}^3 | r = \|\mathbf{x}\| < R\}),\)
2) \(\tilde{\rho}^{\gamma-1}, \tilde{S} \in C^\infty(B_R) \cap C^{2,\alpha}(\overline{B_R})\) with \(0 < \exists \alpha < \left(\frac{\gamma}{\gamma-1} - 2\right) \wedge 1,\)
3) \(d\tilde{\rho}/dr, d\tilde{P}/dr < 0\) for \(0 < r < R,\)

\[\exists \lim_{r \to +0} \frac{1}{r} \frac{d\tilde{\rho}}{dr} =: -\rho_{O1} < 0, \quad \exists \lim_{r \to +0} \frac{1}{r} \frac{d\tilde{P}}{dr} =: -P_{O1} < 0.\]
4) \(-\infty < \left.\frac{d}{dr} \tilde{\rho}^{\gamma-1}\right|_{r=R-0} < 0.\)
Assumption 2. \( \exists \) admissible spherically symmetric equilibrium \((\bar{\rho}, \bar{S})\).

**Fix one of the admissible spherically symmetric equilibria.**

We consider small perturbation from this fixed equilibrium by the Lagrangian co-ordinate system, which will be denoted by the same letters \((t, x^1, x^2, x^3)\) of the Eulerian co-ordinate system. The perturbations \(\xi^j\) of \(x^j\) is defined by

\[
x^j + \xi^j(t, x) = \varphi^j(t, x),
\]

where \(t \mapsto \varphi(t, x) = (\varphi^1(t, x), \varphi^2(t, x), \varphi^3(t, x))\) is the co-ordinate of the stream line, that is, the solution of the initial value problem of the ordinary differential equation

\[
\frac{\partial}{\partial t} \varphi^j(t, x) = v^j(t, \varphi(t, x)), \quad \varphi^j(0, x) = x^j.
\]
It is known that, when the initial perturbation $\rho - \bar{\rho}$ at $t = 0$ vanishes, the linearized approximation of the equations for the perturbations $\xi = \sum \xi^k \partial / \partial x^k$ is
\[
\frac{\partial^2 \xi}{\partial t^2} + L \xi = 0, \tag{6}
\]
where
\[
L \xi = \frac{1}{\bar{\rho}} \text{grad} \delta \rho - \frac{\text{grad} P}{\rho^2} \delta \rho + \text{grad} \delta \Phi, \tag{7}
\]
\[
\delta \rho = -\text{div}(\bar{\rho} \xi), \tag{8}
\]
\[
\delta \Phi = -4\pi G \mathcal{K}(\delta \rho), \tag{9}
\]
\[
\delta P = \frac{\gamma \bar{P}}{\rho} \delta \rho + \gamma A \bar{P}(\xi | e_r). \tag{10}
\]
Here
Definition 2.

\[ A := \frac{1}{\rho} \frac{d\rho}{dr} - \frac{1}{\gamma P} \frac{dP}{dr} \bigg( = - \frac{1}{\gamma C_v} \frac{dS}{dr} \bigg), \]  

(11a)

\[ N^2 := A \frac{1}{\rho} \frac{dP}{dr} = -A \frac{d\Phi}{dr}. \]  

(11b)

Problem: Clarify the spectral properties of the operator $L$ in the Hilbert space $\mathcal{H} := L^2(B_R, \bar{\rho}d\mathbf{x})$.  

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Part I : Isentropic case

Joint work with Juhi Jang (University of Southern California, Korea Institute for Advanced Study)

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2 Barotropic situation

Suppose $S =$ Constant..

$\Rightarrow$

$$P = A \rho^\gamma, \quad A := \exp\left(\frac{S}{C_V}\right) = \text{Const.}$$

$$\mathcal{A} = -\frac{1}{\gamma C_V} \frac{d\tilde{S}}{dr} = 0.$$ 

Put

$$u := \int_0^\rho \frac{dP}{\rho} = \frac{A \gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad (12)$$
If \( \frac{6}{5} < \gamma < 2 \), then \( \exists \) admissible equilibrium for \( \forall \rho_0 > 0 \):

\[
\rho = \bar{\rho} = \rho_0 \theta \left( \frac{r}{a}; \nu \right)^{\nu}, \quad u = \bar{u} = u_0 \theta \left( \frac{r}{a}; \nu \right)
\]

where \( \nu := \frac{1}{\gamma - 1} \), \( a = \sqrt{\frac{A \gamma}{4\pi G(\gamma - 1)}} \rho_0^{-\frac{\gamma-2}{2}} \) and \( \theta(\xi; \nu) \) is the Lane-Emden equation of index \( \nu \):

\[
-\frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \frac{d\theta}{d\xi} = (\theta \vee 0)^{\nu}, \quad \theta = 1 + O(\xi^2) \quad \text{as} \quad \xi = +0
\]

\[
\theta(\xi_1(\nu)) = 0, \quad \mu_1(\nu) = -\xi^2 \frac{d\theta}{d\xi} \bigg|_{\xi=\xi_1(\nu)} < 0.
\]
We are considering

$$\frac{\partial^2 \xi}{\partial t^2} + L\xi = 0,$$

(14)

with

$$L\xi = \text{grad}\left(-\frac{1}{\rho} \frac{dP}{d\rho} g + 4\pi G\kappa g\right),$$

(15)

$$g(= -\delta \rho) = \text{div}(\bar{\rho}\xi),$$

(16)

thanks to $$A = -\frac{1}{\gamma C_V} \frac{d\tilde{S}}{dr} = 0.$$
3 Self-adjoint operator by the Friedrichs extension

We consider the integro-differential operator $L$ acting on the field $\xi$:

$$L\xi = \text{grad} \left( -\frac{1}{\rho} \frac{dP}{d\rho} \text{div}(\rho\xi) + 4\pi G K \text{div}(\rho\xi) \right)$$

(17)

in the Hilbert space $\mathcal{H} = L^2(B_R, \rho dx)$
The operator $\mathbf{L} : C_0^\infty(B_R) : \xi \in C_0^\infty(B_R; \mathbb{C}^3) \mapsto \mathbf{L}\xi$ is symmetric, that is,

$$(\mathbf{L}\xi_1,\xi_2)_{\mathcal{H}} = (\xi_1,\mathbf{L}\xi_2)_{\mathcal{H}}$$

for any $\xi_1, \xi_2 \in C_0^\infty(B_R)$, and is bounded from below, that is, there exists a constant $C$ such that

$$(\mathbf{L}\xi,\xi)_{\mathcal{H}} \geq -C\|\xi\|^2_{\mathcal{H}} \quad \text{for} \quad \forall \xi \in C_0^\infty(B_R).$$

Note

$$(\mathbf{L}\xi,\xi)_{\mathcal{H}} = \int_{B_R} \frac{1}{\rho} \frac{dP}{d\rho} |g|^2 - 4\pi G(Kg)g^* \, d\mathbf{x}$$

(18)
Therefore the operator $\mathbf{L} \upharpoonright C_0^\infty(B_R)$ admits the Friedrichs extension $\mathcal{T}$ which is a self-adjoint operator in the Hilbert space $\mathcal{H}$. The domain $D(\mathcal{T})$ of the operator $\mathcal{T}$ is

$$D(\mathcal{T}) = \{ \xi \in \mathring{\mathcal{F}} \mid \mathbf{L}\xi \in \mathcal{H} \text{ in distribution sense} \}. \quad (19)$$

Here $\mathring{\mathcal{F}}$ is the closure of $C_0^\infty(B_R)$ in the Hilbert space $\mathring{\mathcal{F}} = \{ \xi \in \mathcal{H} \mid \text{div}(\bar{\rho}\xi) \in \mathcal{G} \}$ endowed with the inner product $(\xi_1 | \xi_2)_{\mathring{\mathcal{F}}} = (\xi_1 | \xi_2)_{\mathcal{F}} + (\text{div}(\bar{\rho}\xi_1) | \text{div}(\bar{\rho}\xi_2))_{\mathcal{G}}$, where $\mathcal{G} = L^2(B_R, \frac{1}{\rho} \frac{dP}{d\rho} \, dx) \cap \{ g \mid \int_{B_R} g \, dx = 0 \}$ endowed with the inner product $(g_1 | g_2)_{\mathcal{G}} = \int_{B_R} g_1 g_2^* \frac{1}{\rho} \frac{dP}{d\rho} \, dx$. 

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Hereafter we shall denote by the same letter $L$ the Friedrichs extension $\mathcal{T}$. Thus we can claim the following

**Theorem 1.** The operator $L$ is a self-adjoint operator bounded from below in the Hilbert space $\mathcal{H}$. 
Does the variational principle work?

Since $Q[\xi] = (L\xi|\xi)_{\mathcal{H}}$ is bounded from below, we have the finite infimum

$$\lambda := \inf \{ Q[\xi] \mid \xi \in \mathcal{H}_1, \|\xi\|_{\mathcal{H}} = 1 \}. \quad (20)$$

Is $\lambda$ is an eigenvalue?

Does it hold

‘Rellichscher Auswahlsatz’ (♡) : If $\{ Q[\xi_n] + \kappa \|\xi_n\|_{\mathcal{H}}^2 \}$, where $\kappa = 1 - \lambda$, is bounded, then there is a subsequence converging in $\mathcal{H}$, or, $\mathfrak{F}$ is compactly imbedded in $\mathfrak{H}$. ?
Definition 3. The spectrum $\sigma(T)$ of a self-adjoint operator $T$ in an infinitely dimensional Hilbert space $X$ is said to be of the Sturm-Liouville type if $\sigma(T)$ consists of eigenvalues with finite multiplicities which accumulate to $+\infty$.

The Riesz-Schauder’s theorem: If a resolvent $(\lambda - T)^{-1}$ of the self-adjoint operator $T$ bounded from below is a compact operator, then the spectrum of the operator $T$ is of the Sturm-Liouville type.
Now the operator $L + \kappa$ has the inverse $(L + \kappa)^{-1}$ which is a bounded linear operator on $\mathcal{H}$ into itself with operator norm $\leq 1$.

If $(\heartsuit)$ hold, then $(L + \kappa)^{-1}$ would be a compact operator, so, by the Riesz-Schauder’s theorem implies that the spectrum of $L$ would be of the Sturm-Liouville type.

However $(\heartsuit)$ does not hold, since the functional space

$$\mathcal{N} = \{\xi \in \mathcal{H} \mid \text{div}(\bar{\rho}\xi) = 0 \text{ in distribution sense}\} \quad (21)$$

is a subset of $\text{Ker}L$, that is, any $\xi \in \mathcal{N}, \neq 0$ is an eigenfunction of $L$ associated with the eigenvalue 0, while the dimension of $\mathcal{N}$ is infinite.
In fact, for any vector function \( \mathbf{A} \in C_0^\infty(B_R) \), the vector function
\[
\xi = \frac{1}{\rho} \text{curl} \mathbf{A}
\]
belongs to \( \mathfrak{M} \). This is a contradiction to the assertion that the spectrum of \( \mathbf{L} \) was of the Sturm-Liouville type.

In other words, we cannot expect the ‘Rellichscher Auswahlsatz’ (♡), if we do not limit ourselves to spherically symmetric perturbations or curl-free perturbations. Then it is doubtful that the so called ‘variational principle’ that the minimum \( \lambda \) should be attained at an eigenfunction \( \xi_\infty \).
4 Spectral analysis of an auxiliary operator $N$

Introducing $\hat{\xi} = \bar{\rho}\xi$, the equation (14) reads

\[
\frac{\partial^2 \hat{\xi}}{\partial t^2} + \hat{L}\hat{\xi} = 0,
\]  

(22)

where

\[
\hat{L}\hat{\xi} = \hat{M}g, \quad g = \text{div}\hat{\xi},
\]

\[
\hat{M}g = \rho\text{grad}\left(-\frac{1}{\rho} \frac{dP}{d\rho} g + 4\pi G\kappa g\right).
\]  

(23)
Taking the divergence of (22), we get

\[
\frac{\partial^2 g}{\partial t^2} + \mathcal{N} g = 0, \quad (24)
\]

where

\[
\mathcal{N} g = - \rho \frac{d\rho}{dP} \text{div} \left( \frac{1}{\rho} \left( \frac{dP}{d\rho} \right)^2 \text{grad} g \right) - \left[ \Delta \left( \frac{dP}{d\rho} - u \right) \right] g
\]

\[
+ 4\pi G \text{div} \left( \tilde{\rho} \text{grad} (Kg) \right). \quad (25)
\]

Let us consider the operator \( \mathcal{N} \) in a Hilbert space

\[
\mathcal{G} = L^2 \left( B_R; \frac{1}{\rho} \frac{dP}{d\rho} dx \right) \cap \{ g | \int_{B_R} g dx = 0 \}.
\]

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Let us analyze the operator $\mathcal{N}$, which can be decomposed as

$$\mathcal{N}g = \mathcal{N}_{00}g + \mathcal{N}_{01}g,$$  \hspace{1cm} (26a)

with

$$\mathcal{N}_{00}g = -\rho \frac{d\rho}{dP} \text{div}\left( \frac{1}{\rho} \left( \frac{dP}{d\rho} \right)^2 \text{grad}g \right) - \left[ \Delta \left( \frac{dP}{d\rho} - u \right) \right] g,$$ \hspace{1cm} (26b)

$$\mathcal{N}_{01}g = 4\pi G \text{div}(\bar{\rho} \text{grad}(\mathcal{K}g)).$$ \hspace{1cm} (26c)
Note

\[ Q[g] = (\mathcal{N}_{00}g|g)_\mathcal{G} = \]
\[ = \int \|\text{grad} g\|^2 \frac{1}{\rho} \left( \frac{dP}{d\rho} \right)^2 dx - \int \left[ \triangle \left( \frac{dP}{d\rho} - u \right) \right] |g|^2 \frac{1}{\rho} \frac{dP}{d\rho} dx. \quad (27) \]

The operator \( \mathcal{N}_{00} \upharpoonright C_0^\infty (B_R) : g \in C_0^\infty (B_R, \mathbb{C}) \mapsto \mathcal{N}_{00}g \) is symmetric and bounded from below, since

\[ \left| \triangle \left( \frac{dP}{d\rho} - u \right) \right| \leq C. \]

Moreover we can show the inverse is compact (completely continuous), that is, the closure in \( \mathcal{G} \) of \( \{ g \mid Q[g] + \kappa \|g\|^2_\mathcal{G} \leq B \} \) is compact, \( \kappa \) being sufficiently large.
Actually $Q[g] + \kappa \|g\|_{\mathfrak{S}}^2$, which is equivalent to

$$
\|g\|_{\mathfrak{S}_1}^2 = \int \|\text{grad} g\|^2 \frac{1}{\rho} \left(\frac{dP}{d\rho}\right)^2 dx + \int |g|^2 \frac{1}{\rho} \frac{dP}{d\rho} dx,
$$

(28)

can control $\|\text{grad} g\|_{L^2(...)}^2$, and $\mathfrak{S}_1$, which is the closure of $C^\infty_0(B_R)$ in the Hilbert space $\mathfrak{S}_1$ endowed with the norm $\| \cdot \|_{\mathfrak{S}_1}$, is compactly imbedded into $\mathfrak{S}$.

Here we use the theory of the weighted Sobolev spaces developed by the Czech school.
The operator $\mathcal{N} \upharpoonright C_0^\infty(B_R) : g \in C_0^\infty(B_R, \mathbb{C}) \mapsto \mathcal{N}g = \mathcal{N}_{00}g + \mathcal{N}_{01}g$ has the Friedrichs extension $\mathcal{T}$, which is a self-adjoint operator in $\mathfrak{G}$, and the resolvent $(\mathcal{T} + \kappa)^{-1}$ is a compact operator in $\mathfrak{G}$. The spectrum of $\mathcal{T}$ is of the Sturm-Liouville type.

Hereafter let us use the letter $\mathcal{N}$ for the Friedrichs extension $\mathcal{T}$. Thus we have the following

**Theorem 2.** The operator $\mathcal{N}$ is a self-adjoint operator bounded from below in the Hilbert space $\mathfrak{G}$ and its spectrum $\sigma(\mathcal{N})$ is of the Sturm-Liouville type.
5 Spectral analysis of $L$

Use the variable $\hat{\xi} = \rho \xi$ in the Hilbert space $\hat{\mathcal{H}} = L^2(B_R, \frac{1}{\rho} dx) = \rho \mathcal{H}$.

We put

$$\hat{\mathcal{F}} = \{\hat{\xi} \in \hat{\mathcal{H}} \mid \text{div} \hat{\xi} \in \mathcal{G}\}, \quad (29)$$

defining the inner product

$$(\hat{\xi}_1|\hat{\xi}_2)_{\hat{\mathcal{F}}} = (\hat{\xi}_1|\hat{\xi}_2)_{\hat{\mathcal{H}}} + (\text{div} \hat{\xi}_1|\text{div} \hat{\xi}_2)_{\mathcal{G}}. \quad (30)$$
The operator $\hat{M}$ is a bounded linear operator from $\mathcal{G}_1$ into $\hat{H}$.

\[
D(\hat{L}) = \{ \hat{\xi} \in \hat{\mathcal{F}} | \text{div} \hat{\xi} \in D(\mathcal{N}) \}.
\]
Proposition 1. If $\lambda \in \varrho(\mathcal{N}) \setminus \{0\}$, then the operator $(\lambda - \hat{L})^{-1}$ is a bounded linear operator from $\mathcal{F}$ into $\mathcal{F}$, that is, $\lambda$ belongs to $\varrho(\hat{L})$, the resolvent set of $\hat{L}$.

Proof. Take an arbitrary $\lambda \neq 0$ from the resolvent set $\varrho(\mathcal{N})$ of the operator $\mathcal{N}$ and consider the resolvent $(\lambda - \mathcal{N})^{-1}$, which is a bounded linear operator from $\mathcal{G}$ into $\mathcal{G}_1$. Then the equation

$$(\lambda - \hat{L})\hat{\xi} = \hat{f} \in \mathcal{F}$$

can be solved as

$$\hat{\xi} = \frac{1}{\lambda}(\hat{f} + \hat{M}(\lambda - \mathcal{N})^{-1}\text{div}\hat{f}).$$

In fact, (32) implies

$$\lambda\hat{\xi} = \hat{f} + \hat{M}(\lambda - \mathcal{N})^{-1}\text{div}\hat{f};$$
putting $f = (\lambda - \mathcal{N})^{-1} \text{div} \hat{f}$, we have

$$\text{div} \hat{f} = (\lambda - \mathcal{N}) f, \quad \lambda \hat{\xi} = \hat{f} + \mathcal{M} f$$

so that

$$\lambda \text{div} \hat{\xi} = \text{div} \hat{f} + \text{div} \mathcal{M} f$$

$$= (\lambda - \mathcal{N}) f + \mathcal{N} f = \lambda f;$$

therefore $\text{div} \hat{\xi} = f$, since $\lambda \neq 0$; then we have

$$\lambda \hat{\xi} = \hat{f} + \mathcal{L}(\lambda - \mathcal{N})^{-1} \text{div} \hat{f}$$

$$= \hat{f} + \mathcal{M} f$$

$$= \hat{f} + \mathcal{M} \text{div} \hat{\xi}$$

$$= \hat{f} + \mathcal{L} \hat{\xi},$$
that is (31).

Since we know that the operator $\hat{M}(\lambda - \mathcal{N})^{-1}$ is a bounded linear operator from $\mathcal{G}$ into $\hat{\mathcal{H}}$ and since

$$\|\text{div}\hat{M}(\lambda - \mathcal{N})^{-1}f\|_\mathcal{G} = \|\mathcal{N}(\lambda - \mathcal{N})^{-1}f\|_\mathcal{G}$$

$$\leq \|\|\lambda(\lambda - \mathcal{N})^{-1} - I\|\|f\|_\mathcal{G},$$

the operator $\hat{M}(\lambda - \mathcal{N})^{-1}$ is a bounded operator from $\mathcal{G}$ into $\hat{\mathcal{H}}$. □

**Proposition 2.** The operator $\hat{L}$ considered in the Hilbert space $\hat{\mathcal{H}}$ is self-adjoint.
Proposition 3. A non-zero eigenvalue of $\mathcal{N}$ is an eigenvalue of $\hat{\mathcal{L}}$.

Proposition 4. If $\lambda \in \sigma(\hat{\mathcal{L}})$ and if $\lambda \neq 0$, then $\lambda \in \sigma(\mathcal{N})$ and $\lambda$ is an eigenvalue of the operator $\hat{\mathcal{L}}$ of finite multiplicity.

Proof. Let $\lambda \in \sigma(\hat{\mathcal{L}}) \setminus \{0\}$. Then

$$\hat{\xi} \in \ker(\lambda - \hat{\mathcal{L}}) \iff \hat{\xi} = \frac{1}{\lambda}\hat{M}g \quad \text{with} \quad g \in \ker(\lambda - \mathcal{N}).$$

Therefore

$$\dim \ker(\lambda - \hat{\mathcal{L}}) \leq \dim \ker(\lambda - \mathcal{N}).$$

□

Summing up, we have
The operator $\hat{L}$ is a self-adjoint operator in $\hat{\mathcal{F}}$. Its spectrum $\sigma(\hat{L})$ coincides with $\sigma(\mathcal{N}) \cup \{0\}$, while $\dim \text{Ker}(\hat{L}) = \infty$ and $\lambda \in \sigma(\hat{L}) \setminus \{0\}$ is an eigenvalue of finite multiplicity.

Translating this statement to that in terms in $L$, we can claim the following

**Theorem 3.** The operator $L$ is a self-adjoint operator in $\mathcal{F}$. Its spectrum $\sigma(L)$ coincides with $\sigma(\mathcal{N}) \cup \{0\}$, while $\dim \text{Ker}(L) = \infty$ and $\lambda \in \sigma(L) \setminus \{0\}$ is an eigenvalue of finite multiplicity.

Here the Hilbert space $\mathcal{F}$ is nothing but $\frac{1}{\rho} \hat{\mathcal{F}}$ endowed with the inner product

$$ (\xi_1|\xi_2)_{\mathcal{F}} = (\xi_1|\xi_2)_{\mathcal{F}} + (\text{div}(\rho \xi_1)|\text{div}(\rho \xi_2))_{\mathcal{G}}. $$
Remark 1. Here $L$ stands for the Friedrichs extension of $L \upharpoonright C_0^\infty(B_R)$ in $\mathcal{F}$, and is different from the Friedrichs extension considered in Section 2, which was the Friedrichs extension of $L \upharpoonright C_0^\infty(B_R)$ not in $\mathcal{F}$ but in $\mathcal{F}'$.

Open Problem 1. How about the spectrum $\sigma(L)$ of the operator $L$ considered in $\mathcal{H} = L^2(B_R, \bar{p}dx)$? For $\lambda \neq 0, \lambda \notin \sigma(N)$ can we claim that $(\lambda - L)^{-1}$ is bounded re $\mathcal{H}$-norm so that $\lambda \in \varrho(L)$?
Part II: General Adiabatic Case

arXiv: 1902.03675
6 Existence of admissible spherically symmetric equilibrium

Theorem 4. Let $\Sigma : \eta \mapsto \Sigma(\eta)$ which belongs to $C^\infty(\mathbb{R})$ and $\rho_0 > 0$ be given. Assume

$$\gamma + \frac{\gamma - 1}{C_V} \eta \frac{d\Sigma}{d\eta} > 0 \quad \text{for} \quad \eta > 0.$$ 

If $\frac{4}{3} < \gamma < 2$ or if $\frac{6}{5} < \gamma \leq \frac{4}{3}$ and $\rho_0 \ll 1$, then $\exists$ admissible spherically symmetric equilibrium $(\tilde{\rho}, \tilde{S})$ such that $\tilde{S} = \Sigma(\tilde{\rho}^{\gamma-1}), \tilde{\rho}(O) = \rho_0$. 
Proposition 5. Let $\bar{S} = \Sigma(\bar{\rho}^{\gamma-1})$. If

$$\frac{d\Sigma}{d\eta} < 0 \quad \text{for} \quad \eta > 0,$$

then $A < 0$ for $0 < r \leq R$. 
7 Self-adjoint realization of $\mathbf{L}$

\[
\mathbf{L}\xi = \frac{1}{\rho}\text{grad}\delta P - \frac{\text{grad}P}{\rho^2}\delta\rho + \text{grad}\delta\Phi,
\]
\[
\delta\rho = -\text{div}(\bar{\rho}\xi),
\]
\[
\delta P = \frac{\gamma P}{\rho}\delta\rho + \gamma A\bar{P}(\xi|e_r),
\]
\[
\delta\Phi = -4\pi G\mathcal{K}(\delta\rho).
\]

**Theorem 5.** $\mathbf{L} \upharpoonright C_0^\infty(B_R)$ admits the Friedrichs extension, which is a self-adjoint operator bounded from below, in the Hilbert space $\mathcal{H} = L^2(B_R, \bar{\rho}dx)$. 


• $\dim N(L) = \infty$.

**Open Problem 2.** Let $A < 0$ everywhere on $0 < r < R$. Maybe it does not hold that

$$\sigma(L) = \{0\} \cup \{\lambda_n\}, \quad \dim N(\lambda_n - L) < \infty,$$

even if we restrict $L$ to some suitable $\mathcal{F} \subset \mathcal{H}$. 

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8  Eigenfunctions represented by spherical harmonics

\( \xi = V^r(r)Y_{lm}(\vartheta, \phi)e_r + V^h(r)\nabla_s Y_{lm}(\vartheta, \phi). \)  \( (33) \)

Here \( l, m \in \mathbb{Z}, 0 \leq l, |m| \leq l, \) and \( Y_{lm} \) is the spherical harmonics:

\[
Y_{lm}(\vartheta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \vartheta)e^{\sqrt{-1}m\phi},
\]

\( Y_{l,-m} = (-1)^m Y_{lm}^* \) for \( m \geq 0. \)
\[x^1 = r \sin \vartheta \cos \phi,\]
\[x^2 = r \sin \vartheta \sin \phi,\]
\[x^3 = r \cos \vartheta.\]

\[\nabla_s f := \frac{\partial f}{\partial \vartheta} \mathbf{e}_\vartheta + \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi\]

\[\delta \rho = \delta \rho(r) Y_{lm}(\vartheta, \phi), \quad (34a)\]
\[\delta P = \delta P(r) Y_{lm}(\vartheta, \phi), \quad (34b)\]
\[\delta \Phi = \delta \Phi(r) Y_{lm}(\vartheta, \phi), \quad (34c)\]
where

\[ \delta \ddot{\rho} = -\frac{1}{r^2} \frac{d}{dr} (r^2 \rho V^r) + \frac{l(l + 1)}{r} \rho V^h, \]  

(35a)

\[ \delta \ddot{P} = \frac{\gamma P}{\rho} \delta \dot{\rho} + \gamma A PV^r \]

\[ = -\frac{\gamma P}{r^2 \rho} \frac{d}{dr} (r^2 \rho V^r) + \gamma A PV^r + l(l + 1) \frac{\gamma P}{r} V^h, \]  

(35b)

\[ \delta \ddot{\Phi} = -4\pi G \mathcal{H}_l (\delta \dot{\rho}) \]

\[ = 4\pi G \mathcal{H}_l \left( \frac{1}{r^2} \frac{d}{dr} (r^2 \rho V^r) - \frac{l(l + 1)}{r} \rho V^h \right). \]  

(35c)
Here the integral operator $\mathcal{H}_l$, which solves

$$\left[ -\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{l(l+1)}{r^2} \right] \mathcal{H}_l f = f,$$

is defined by

$$\mathcal{H}_l f(r) = \frac{1}{2l+1} \left[ \int_r^\infty f(r') \left( \frac{r}{r'} \right)^l r' dr' + \int_0^r f(r') \left( \frac{r}{r'} \right)^{-(l+1)} r' dr' \right],$$

(36)
We mean
\[ \mathbf{L}(V^r Y_{lm} e_r + V^h \nabla_s Y_{lm}) = L^r_l Y_{lm} e_r + L^h_l \nabla_s Y_{lm}. \] (37)

Note \( Y_{00} = \frac{1}{\sqrt{4\pi}} \) so that we can forget \( L^h_0 \) and the component \( V^h \) for \( l = 0 \).

The wave equation reads
\[
\frac{\partial^2 V^r}{\partial t^2} + L^r_l = 0, \quad \frac{\partial^2 V^h}{\partial t^2} + L^h_l = 0,
\] (38)
where

\begin{align}
L_l^r &= \frac{1}{\rho} \frac{d}{dr} \delta \tilde{P} - \frac{1}{\rho^2} \frac{dP}{dr} \delta \tilde{\rho} + \frac{d}{dr} \delta \tilde{\Phi}, \\
L_l^h &= \frac{1}{r} \left( \frac{\delta \tilde{P}}{\rho} + \delta \tilde{\Phi} \right). 
\end{align}

We are going to analyze the operator \( \tilde{L}_l = \begin{bmatrix} L_l^r \\ L_l^h \end{bmatrix} \) which acts on

\[ \tilde{V} = \begin{bmatrix} V^r \\ V^h \end{bmatrix}. \]
8.1 Case \( l = 0 \)

**Theorem 6.** The operator \( L^{ss} \) on \( C_0^\infty(]0, R[) \) admits the Friedrichs extension, a self-adjoint operator bounded from below in \( L^2([0, R], \bar{\rho}r^4dr) \), and its spectrum consists of simple eigenvalues \( \lambda_1^{ss} < \lambda_2^{ss} < \cdots < \lambda_n^{ss} < \cdots \to +\infty \).

Here

\[
L^{ss}\psi = \frac{1}{r}L'_0((r\psi, 0)^\top),
\]

while \( L_0^h \) need not be considered.
8.2 Case \( l \geq 1 \)

Suppose \( l \geq 1 \).

Let us consider the Hilbert space \( \mathcal{X}_l \) of functions \( \vec{f} = (f^r, f^h) \) defined on \([0, R[\) endowed with the norm \( \| \vec{f} \|_{\mathcal{X}_l} \) given by

\[
\| \vec{f} \|_{\mathcal{X}_l}^2 = \int_0^R (|f^r(r)|^2 + l(l+1)|f^h(r)|^2) \rho r^2 dr.
\]  

(40)

Note

\[
\| \xi \|_{\mathcal{V}} = \sqrt{4\pi} \| \vec{V} \|_{\mathcal{X}_l}
\]

under (33).
**Theorem 7.** The integro-differential operator $\vec{L}_l$ on $C_0^\infty([0, R[, C^2)$ admits the Friedrichs extension, which is a self-adjoint operator bounded from below, in $\mathfrak{X}_l$.

Note

$$(\vec{L}_l\vec{V}|\vec{V})_{\mathfrak{X}_l} = \gamma \int |W|^2 dr + \int A \frac{dP}{dr} |V^r|^2 r^2 dr - 4\pi G \int H_l(\delta \tilde{P})(\delta \tilde{P}^*) r^2 dr,$$

where

$$W := \frac{P^{\frac{1}{2} - \frac{1}{\gamma}}}{r} \frac{d}{dr} (r^2 P^{\frac{1}{\gamma}} V^r) - l(l + 1) P^{\frac{1}{2}} V^h$$

$$= -\frac{1}{\gamma} r P^{-\frac{1}{2}} \delta \tilde{P}.$$
Note

\[ D(\vec{L}_l) = \mathcal{M}_l \cap \{ \vec{V} \mid \vec{L}_l \vec{V} \in \mathcal{X}_l \text{ in distribution sense} \}. \quad (41) \]

Here \( \mathcal{M}_l \) is the closure of \( C^\infty_0([0, R[, \mathbb{C}^2) \) in the Hilbert space \( \mathcal{M}_l \) endowed with the norm \( \| \cdot \|_{\mathcal{M}_l} \) given by

\[
\| \vec{V} \|^2_{\mathcal{M}_l} = \| \vec{V} \|^2_{\mathcal{X}_l} + \| \delta \hat{\rho} \|^2_{L^2(\frac{\gamma P}{\rho^2} r^2 dr)}, \quad (42)
\]

where

\[
\| \delta \hat{\rho} \|^2_{L^2(\frac{\gamma P}{\rho^2} r^2 dr)} = \int_0^R \left| -\frac{1}{r^2} \frac{d}{dr}(r^2 \rho V_r) + \frac{l(l+1)}{r} \rho V^h \right|^2 \frac{\gamma P}{\rho^2} r^2 dr. \quad (43)
\]

**Proposition 6.** If \( \vec{V} \in \mathcal{M}_l \) satisfies \( |V^r| \leq C \), then \( \vec{V} \) belongs to \( \mathcal{M}_l \).
Theorem 8. 1) Let $l \geq 1$. Suppose that $A = 0$ identically on $]0, R[$. Then $\dim N(\vec{L}_l) = \infty$.

2) Suppose that $A < 0$ everywhere on $]0, R[$. Then $\dim N(\vec{L}_l) = 0$ when $l \geq 2$ and $\dim N(\vec{L}_1) = 1$ when $l = 1$.

Open Problem 3. Let $A < 0$ everywhere on $0 < r < R$. Then maybe it does not hold that

$$
\sigma(\vec{L}_l) = \{0\} \cup \{\lambda^n[l], n \in \mathbb{N}\}, \quad \dim N(\lambda^n[l] - \vec{L}_l) < \infty,
$$

which is the case if $A = -\frac{1}{\gamma C_V} \frac{d\tilde{S}}{dr} = 0$. 
9 Cowling approximation, g-modes, p-modes

Let us consider the case in which \( l \geq 1, \lambda \neq 0 \).

The Cowling approximation:

\[
\vec{L}_{0l} \vec{V} = \lambda \vec{V},
\]

where

\[
\vec{L}_{0l} = \begin{bmatrix}
L^{r}_{0l} \\
L^{h}_{0l} \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\rho} \frac{d}{dr} \delta \tilde{P} - \frac{1}{\rho^2} \frac{dP}{dr} \delta \tilde{\rho} \\
1 \frac{\delta \tilde{P}}{r \rho} \\
\end{bmatrix}.
\]
Introduce the variables

\[ v = r^2 P^{\frac{1}{\gamma}} V^r, \quad w = P^{-\frac{1}{\gamma}} \delta \tilde{P}. \]  \hspace{1cm} (46)

by which

\[ V^r = \frac{1}{r^2} P^{-\frac{1}{\gamma}} v, \]  \hspace{1cm} (47a)

\[ V^h = \frac{1}{l(l+1)} \left[ \frac{1}{r} P^{-\frac{1}{\gamma}} \frac{dv}{dr} + \frac{1}{rP^{-1+\frac{1}{\gamma}}} w \right], \]  \hspace{1cm} (47b)

\[ \delta \tilde{P} = P^{\frac{1}{\gamma}} w, \]  \hspace{1cm} (47c)

\[ \delta \tilde{\rho} = \frac{1}{\gamma} \rho P^{-1+\frac{1}{\gamma}} w - \frac{1}{r^2} \mathcal{A} \rho P^{-\frac{1}{\gamma}} v, \]  \hspace{1cm} (47d)

\[ \frac{d}{dr} \delta \tilde{P} - \frac{1}{\rho} \frac{dP}{dr} \delta \tilde{\rho} = P^{\frac{1}{\gamma}} \frac{dw}{dr} + \frac{1}{r^2} \mathcal{A} P^{-\frac{1}{\gamma}} \frac{dP}{dr} v. \]  \hspace{1cm} (47e)
The eigenvalue problem (44) reads

\[
\frac{dv}{dr} + \frac{r^2 \rho}{\gamma P} B w = \frac{1}{\lambda} l(l + 1) B w, \tag{48a}
\]

\[
\frac{dw}{dr} + \frac{\mathcal{N}^2}{r^2 B} v = \lambda \frac{1}{r^2 B} v. \tag{48b}
\]

Here

\[
B := \frac{P_{\gamma}^2}{\rho}, \quad \mathcal{N}^2 = \frac{A}{\rho} \frac{dP}{dr}. \tag{49}
\]
Assumption 3. There is a positive number $C$ such that it holds that

\[ \frac{1}{C^r} \leq \frac{d\tilde{S}}{dr} \]  \hspace{1cm} (50)

for $0 < r \leq R$. 
[g-modes]

Put $\lambda = 0$ in (48b):

$$\frac{dw}{dr} + \frac{N^2}{r^2 B} v = 0.$$  

Then (48a) turns to be

$$L^{[g]}_l w = \frac{1}{\lambda} w, \quad (51)$$

where

$$L^{[g]}_l w = -\frac{1}{l(l+1)B} \frac{d}{dr} \left( \frac{r^2 B}{N^2} \frac{dw}{dr} \right) + \frac{r^2 \rho}{l(l+1)\gamma P} w. \quad (52)$$
Theorem 9. The operator $L^{[g]}_l$ defined on $C_0^\infty([0, R])$ admits the Friedrichs extension, a self-adjoint operator in $L^2([0, R], l(l + 1)Bdr)$, whose spectrum consists of simple or double eigenvalues $\frac{1}{\lambda_n^g}, n = 1, 2, \cdots : \frac{1}{\lambda_1^g} \leq \frac{1}{\lambda_2^g} \leq \cdots \leq \frac{1}{\lambda_n^g} \rightarrow +\infty$. The eigenvalues are simple if $\gamma > 3/2$. 
[p-modes]

Put $\lambda = \infty$ at (48a). Then (48b) turns out to be

$$L_l^{[p]} v = \lambda v, \quad L_l^{[p]} v = -r^2B \frac{d}{dr}\left( \frac{\gamma P}{r^2 \rho B} \frac{dv}{dr} \right) + N^2v.$$ 

**Theorem 10.** The operator $L_l^{[p]}$ defined on $C^\infty_0(0, R]$ admits the Friedrichs extension, a self-adjoint operator in $L^2([0, R], \frac{1}{r^2 B} dr)$, whose spectrum consists of simple eigenvalues $\lambda_n^p, n = 1, 2, \cdots, : \lambda_1^p < \lambda_2^p < \cdots < \lambda_n^p < \cdots \to +\infty.$
Open Problem 4. Let $l \geq 1$. We have had two sequences of eigenvalues $\lambda_{-n}^{[l]} = \lambda_n^g, \lambda_n^{[l]} = \lambda_n^p$, such that $\lambda_{n}^{[l]}$ tends to $+0$ as $\nu \to -\infty$ and to $+\infty$ as $\nu \to +\infty$. Can they give a good approximation for the eigenvalues of $\vec{L}_{0l}$?

Maybe justification of the Cowling approximation should be relatively easy.
THANK YOU

FOR YOUR ATTENTION!

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