On spherically symmetric solutions of the Einstein-Euler equations

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Abstract

We construct spherically symmetric solutions to the Einstein-Euler equations, which give models of gaseous stars in the framework of the general theory of relativity. We assume a realistic barotropic equation of state. Equilibria of the spherically symmetric Einstein-Euler equations are given by the Tolman-Oppenheimer-Volkoff equations, and time periodic solutions around the equilibrium of the linearized equations can be considered. Our aim is to find true solutions near these time-periodic approximations. Solutions satisfying so called physical boundary condition at the free boundary with the vacuum will be constructed using the Nash-Moser theorem. This work also can be considered as a touchstone in order to estimate the universality of the method which was originally developed for the non-relativistic Euler-Poisson equations.

Key Words and Phrases. Einstein equations, Spherically symmetric solutions, Vacuum boundary, Nash-Moser theorem

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1 Introduction

Recently U. Brauer and L. Karp [2] established a local existence theorem of solutions to the Cauchy problem for the Einstein-Euler equations, which describes a relativistic self-gravitating perfect fluid having density either compactly supported or falling off at infinity in an appropriate manner.

In their work [2] the energy-momentum tensor of the perfect fluid takes the form

\[ T^{\mu\nu} = (\epsilon + P)U^\mu U^\nu - P g^{\mu\nu}, \]

where \( \epsilon = c^2 \rho \) is the energy density, \( P \) is the pressure, and \( U^\mu \) is the four-velocity vector. Here it is assumed that \( P = K \epsilon^\gamma, K > 0, \gamma > 1 \), and the quantity

\[ w := \epsilon^{\frac{\gamma-2}{2}} = c^{\gamma-1} \rho^{\frac{\gamma-2}{2}} \]

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is introduced. The main result requires that the initial data satisfy \( w \in H_{s+1} \) with \( s > 3/2 \) so that \( w \in C^1 \) at least.

However a spherically symmetric equilibrium, which solves the Tolman-Oppenheimer-Volkoff equation, satisfies \( w \sim \text{Const.}(r_+ - r)^{1/2} \) as \( r \to r_+ - 0 \) provided that the equilibrium has a finite radius \( r_+ \). See §3. Hence such an equilibrium is excluded from the class of density distributions admissible to this local existence theorem. We are faced with the same situation in the non-relativistic problem governed by the Euler-Poisson equations as discussed in [6].

Recently this trouble was partially overcome by [9] in the Euler-Poisson equations for the non-relativistic case. So, a similar discussion is required for the relativistic problem. It is the aim of this article.

## 2 Spherically symmetric evolution equations

The Einstein equations read

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4} T_{\mu\nu}.
\]

([5, (95.5)]). Here \( R_{\mu\nu} \) is the Ricci tensor and \( R \) is the scalar curvature \( g^{\alpha\beta}R_{\alpha\beta} \) associated with the metric

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu,
\]

and \( T^{\mu\nu} \) is the energy-momentum tensor of the matter. \( G \) is the constant of gravitation \((6.67 \times 10^{-8} \text{cm}^3/\text{g} \cdot \text{sec}^2)\), and \( c \) is the speed of light \((3.00 \times 10^{10} \text{cm/sec})\). The Einstein equations (1) imply the Euler equations

\[
\nabla_\nu T^{\mu\nu} = 0,
\]

where \( \nabla \) denotes the covariant derivative associated with the metric (2.2). The details can be found in [5] or [11].

The energy-momentum tensor of a perfect fluid is given by

\[
T^{\mu\nu} = (c^2 \rho + P)U^\mu U^\nu - Pg^{\mu\nu},
\]

([5, 94.4]), where \( \rho \) is the mass density, \( P \) is the pressure, and \( U^\mu \) stands for the 4-dimensional velocity vector such that \( U^\mu U_\mu = 1 \). In this article we always assume

\[
(A0) \quad \text{P is a given analytic function of } \rho > 0 \text{ such that } 0 < P, 0 < dP/d\rho < c^2 \text{ for } \rho > 0 \text{ and } P \to 0 \text{ as } \rho \to +0.
\]

If we assume the spherical symmetry, the Einstein-Euler equations are reduced as follows.
We consider the metric of the form
\[ ds^2 = e^{2F} c^2 dt^2 - e^{2H} dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \] (2.5)
where \( F, H \) and \( R \) are functions of \( t, r(\geq 0) \). (Here \( R \) does not mean the scalar curvature \( g^{\mu\nu} R_{\mu\nu} \).) Then the non-zero components of the Einstein tensor \( G^\mu_\nu := R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R \), where \( R \) is the scalar curvature, are:

\[
G^0_0 = e^{-2H} \left( -\frac{R^2}{R^2} + 2 \frac{R'}{R} + 2 \frac{H'R'}{R} \right) + e^{-2F} \left( \frac{R^2}{R^2} + 2 \frac{H'\hat{R}}{R} \right) + \frac{1}{R^2},
\]
\[
G^1_1 = e^{-2F} \left( \frac{R^2}{R^2} + 2 \frac{R'}{R} - 2 \frac{F'\hat{R}}{R} \right) - e^{-2H} \left( \frac{R^2}{R^2} + 2 \frac{F'R'}{R^2} \right) + \frac{1}{R^2},
\]
\[
G^2_2 = G^3_3 = e^{-2H} \left( -\frac{R'}{R} + F' - F'^2 + H'F' + \frac{H'R'}{R} - \frac{F'R'}{R} \right) +
\]
\[+ e^{-2F} \left( \frac{R'}{R} + H + H^2 - H\hat{F} + \frac{H\hat{R}}{R} - \frac{F\hat{R}}{R} \right),
\]
\[
e^{2H} G^0_1 = -e^{-2F} G^1_0 = 2 \left( \frac{R'}{R} - \frac{H'R'}{R} - \frac{F'R'}{R} \right).
\]

(5, p.305, (2)(3)(4)(5)). Here \( \hat{A} \) stands for \( \partial A/c \partial t \) and \( A' \) stands for \( \partial A/\partial r \). Of course the coordinates \( x^\mu \) are taken as

\[ x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi. \]

By a freedom of choice of \( r \) we take it in such a way that the flow is apparently static, say, we suppose

\[ U^0 = e^{-F}, \quad U^1 = U^2 = U^3 = 0. \] (2.6)

Then the energy-momentum tensor turns out to be

\[ T^0_0 = c^2 \rho, \quad T^1_1 = T^2_2 = T^3_3 = -P, \quad T^0_1 = T^0_2 = 0. \] (2.7)

The equation \( \nabla_\mu T^\mu_0 = 0 \) gives

\[ c^2 \rho + \left( \frac{H}{R} + \frac{2\hat{R}}{R} \right) (c^2 \rho + P) = 0, \] (2.8)

and the equation \( \nabla_\mu T^\mu_1 = 0 \) gives

\[ P' + F'(c^2 \rho + P) = 0. \] (2.9)

By integrating (2.9) we can suppose that \( F \) is a function of \( \rho \) given by

\[ F = F(\rho) = -\int^\rho \frac{1}{c^2 \rho + P} \frac{dP}{d\rho} d\rho. \] (2.10)

Let us introduce the variable \( m \) by

\[ m = 4\pi \int_0^R \rho R^2 dR = 4\pi \int_0^r \rho R^2 R'dr. \] (2.11)
The variable $V$ is defined by

$$V = c e^{-F} \dot{R}. \quad (2.12)$$

Then the equation $G_0^1 = 0$ turns out to be

$$\dot{H} = \frac{1}{c} e^{F} \frac{V'}{R}. \quad (2.13)$$

Substituting (2.12)(2.13) into (2.8), we have

$$c^2 \dot{\rho} = -\frac{1}{c} e^{F} (c^2 \rho + P) \left( \frac{V'}{R} + 2 \frac{V}{R} \right). \quad (2.14)$$

Eliminating the time derivatives from the equation $G_0^0 = \frac{8 \pi G}{c^2} \rho$, we have

$$\frac{8 \pi G}{c^2} \rho R^2 R' = \left( -RR^2 e^{-2H} + \frac{1}{c^2} RV^2 + R \right)' .$$

Integrating this, keeping in mind that $R$ should vanish at $r = 0$, we get

$$m = \frac{c^2 R}{2G} \left( \frac{V^2}{c^2} + 1 - R^2 e^{-2H} \right), \quad (2.15)$$

from which we get

$$e^{2H} = \left( 1 + \frac{V^2}{c^2} - \frac{2GM}{c^2 R} \right)^{-1} R^2 . \quad (2.16)$$

Differentiating (2.12) with respect to $t$ and using the equation $G_1^1 = -\frac{8 \pi G}{c^4} P$ and (2.15), we obtain

$$\dot{V} = -\frac{GR}{c^2} \left( \frac{m}{R^3} + \frac{4 \pi P}{c^2} \right) - e^{-2H} \frac{R'}{c^2 \rho + P}, \quad (2.17)$$

or, from (2.16),

$$e^{-F} \dot{V} = -GR \left( \frac{m}{R^3} + \frac{4 \pi P}{c^2} \right) - \left( 1 + \frac{V^2}{c^2} - \frac{2GM}{c^2 R} \right) \frac{P'}{R' (\rho + P/c^2)} . \quad (2.18)$$

Differentiating (2.15) with respect to $t$ and using the equation $G_0^0 = 0$, we have

$$\dot{m} e^{-F} = -\frac{4 \pi R^2}{c^3} PV . \quad (2.19)$$

Now the equations (2.12)(2.14)(2.18)(2.19) govern the evolution of unknowns $R, H, \rho, V, m$. The system of equations to be studied is:

$$e^{-F} \dot{R} = V \quad (2.20a)$$

$$e^{-F} \dot{\rho} = -(\rho + P/c^2) \left( \frac{V'}{R} + \frac{2V}{R} \right) \quad (2.20b)$$

$$e^{-F} \dot{V} = -GR \left( \frac{m}{R^3} + \frac{4 \pi P}{c^2} \right) - \left( 1 + \frac{V^2}{c^2} - \frac{2GM}{c^2 R} \right) \frac{P'}{R' (\rho + P/c^2)} \quad (2.20c)$$

$$e^{-F} \dot{m} = -\frac{4 \pi}{c^3} R^2 PV \quad (2.20d)$$
Of course we assume (2.10) and (2.11). The above equations were derived by [10]. The equations (2.20a), (2.20b), (2.20c), (2.20d) are none other than (1.12-R), (8.11), (1.12-U), (1.12-m) of [10] respectively.

The system of coordinates \((t,r)\) is a co-moving Lagrangian system of coordinates moving at each point with the fluid. Therefore if \(t > 0\) for \(0 \leq r < r_+\) and \(\rho = 0\) for \(r_+ \leq r\) at \(t = 0\), then it remains so for all small \(t > 0\) along the time evolution as long as the \(C^1\) solution exists, while the surface radius \(r_+\) is constant. (Of course the value of \(R\) at the surface can change in time.) Especially we have \(m = m_+\) is constant at \(r = r_+\) for all \(t > 0\). Hence we can take \((t,m)\) as another system of co-moving Lagrangian coordinates. Then we have the formula

\[
\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_m - \frac{4\pi}{c^2} e^F R^2 PV \frac{\partial}{\partial m},
\]

(2.21)

and

\[
\frac{\partial}{\partial r} = 4\pi \rho R^2 \frac{\partial}{\partial m}.
\]

(2.22)

Here \(\partial / \partial t\)_r stands for the partial derivative with respect to \(t\) keeping \(r\) constant, and \(\partial / \partial t\)_m stands for that keeping \(m\) constant.

Note that

\[
\frac{\partial R}{\partial m} = \frac{1}{4\pi \rho R^2},
\]

(2.23)

and

\[
\rho = \left(4\pi R^2 \frac{\partial R}{\partial m}\right)^{-1}.
\]

(2.24)

Thus (2.20a) reads

\[
e^{-F} \left(\frac{\partial R}{\partial t}\right)_m = \left(1 + \frac{P}{c^2 \rho}\right) V,
\]

(2.25)

and the equation (2.20d) reads

\[
e^{-F} \left(\frac{\partial V}{\partial t}\right)_m = \frac{4\pi}{c^2} R^2 PV \frac{\partial V}{\partial m} - GR \left(\frac{m}{R^3} + \frac{4\pi P}{c^2}\right) + \left(1 + \frac{V^2}{c^2} - 2Gm \frac{P}{c^2 R}\right)^{-1} \cdot 4\pi R^2 \frac{\partial P}{\partial m},
\]

(2.26)

where we have used the relation

\[
\frac{P'}{R'} = 4\pi \rho R^2 \frac{\partial P}{\partial m},
\]

which comes from (2.22).

Summing up the system of equations (2.25)(2.26) should be solved, while \(\rho, P = P(\rho)\) are given function of \(R^2 \partial R/\partial m\) through (2.24). Moreover under the assumption (A1) specified in the next section, we can put

\[
F = \frac{-u}{c^2} + F(0)
\]

(2.27)
in order to fix the idea, where $F(0)$ is a constant and

$$u = \int_0^\rho \frac{1}{\rho + P(\rho)/c^2} \frac{dP}{d\rho}$$

is a given function of $R^2 \partial R/\partial m$, too. See (2.10). Hence the unknown functions are only $(t, m) \mapsto R$ and $(t, m) \mapsto V$.

The system of equations (2.25)(2.26) will be called $(E_c)$:

$$e^{-F} \frac{\partial R}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right) V,$$

$$e^{-F} \frac{\partial V}{\partial t} = \frac{4\pi}{c^2} R^2 PV \frac{\partial V}{\partial m} - G R \left(\frac{m}{R^2} + \frac{4\pi P}{c^2}\right) + \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R}(1 + \frac{P}{c^2 \rho})^{-1} \cdot 4\pi R^2 \frac{\partial P}{\partial m}\right).$$

Here we have written $\frac{\partial R}{\partial t}, \frac{\partial V}{\partial t}$ simply instead of $(\frac{\partial R}{\partial t})_m, (\frac{\partial V}{\partial t})_m$. The non-relativistic limit as $c \to +\infty$ is of course $(E_\infty)$:

$$\frac{\partial R}{\partial t} = V,$$

$$\frac{\partial V}{\partial t} = -\frac{G m}{R^2} - 4\pi R^2 \frac{\partial P}{\partial m},$$

which is reduced to the second-order single equation [9, (4)], where $g_0, r$ stand for $G, R$.

**Supplementary Remark 1.** The function $F = F(t, r)$ in the components of the metric (2.5) should satisfy (2.9). Therefore, generally speaking, the formula (2.27) should read

$$F = -\frac{u}{c^2} + F_+(t),$$

where $F_+(t)$ is an arbitrary smooth function of $t$, being constant with respect to $r$, or,

$$e^{2F} = C(t)^2 \kappa e^{-2u/c^2},$$

where $\kappa$ is a positive constant which will be specified in the next section, (see (3.9)), and $C(t)$ is an arbitrary positive smooth function of $t$. Then the left-hand sides of (2.25), (2.26) or $(E_c)$ should be interpreted with

$$e^{-F} \frac{\partial}{\partial t} = \frac{1}{C(t)} \frac{1}{\sqrt{\kappa}} e^{u/c^2} \frac{\partial}{\partial t}.$$
instead of \( t \), that is, we specify
\[
e^F = \sqrt{k} \exp \left( -\frac{u}{c^2} \right),
\] (2.28)
without loss of generality.

In this sense, if we are allowed to forestall the discussion, we should say that, in order to fix the idea, the definitions of \( J, H_1, H_2 \) in the following Section 6 (see (6.8a), (6.8b) and (6.9)) should be done by using (2.28), where \( u \) is a given function of \( \rho \) given by
\[
\rho = \rho (1 + y)^{-2} \left( 1 + y + r \frac{\partial y}{\partial r} \right)^{-1}.
\] (4.5)

3 Equilibrium configurations

Let us consider a solution of (2.20a)-(2.20d) which is independent of \( t \), that is,
\[
F = F(\rho(r)), H = H(r), \rho = \rho(r), P = P(\rho(r)), V \equiv 0, R \equiv r.
\]
Then the system of equations (2.20a)-(2.20d) are reduced to
\[
0 = Gr^3 \left( \frac{m}{c^2} + \frac{4\pi P}{c^2} \right) + (1 - \frac{2Gm}{c^2}) \frac{P'}{\rho + P/c^2}.
\]
Therefore the equation to be studied is
\[
\frac{dm}{dr} = 4\pi r^2 \rho, \quad (3.1a)
\]
\[
\frac{dP}{dr} = -(\rho + P/c^2) \frac{G(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/c^2)}, \quad (3.1b)
\]
This equation was first derived by Oppenheimer-Volkoff [12] in 1939.

Let us observe solutions of the Tolman-Oppenheimer-Volkoff equation (3.1). We assume (A0).

**Proposition 1** Let \( \rho_c (> 0) \) and \( P_c = P(\rho_c) \) be given central density and central pressure. Then there is a unique local solution \((m(r), P(r))\), \( 0 \leq r \leq \delta, \delta \) of (3.1), \( \delta \) being a small positive number, such that \( m = 0, P = P_c \) at \( r = 0 \). Moreover we have
\[
m = \frac{4\pi}{3} \rho_cr^3 + O(r^5),
\]
\[
P = P_c - (\rho_c + P_c/c^2)G(4\pi \rho_c/3 + 4\pi P_c/c^2) \frac{r^2}{2} + O(r^4)
\]
as \( r \to 0 \).

A proof can be found in [7].

We consider the domain of the equation (3.1) as \( D := \{(r, m, P) \mid 0 < r < +\infty, 0 < P < +\infty, 0 < 2Gm/c^2r < 1\} \). Prolonging the local solution as long as possible in the domain \( D \), we have \((0, r_+)\) the maximal interval of existence. Here \( r_+ \leq +\infty \) is a constant.
Definition 1 If \( r_+ = +\infty \), the solution will be called a long equilibrium with the central density \( \rho_c \). If \( r_+ < +\infty \), the solution will be called a short equilibrium.

Remark. It will be shown that, if \( r_+ < +\infty \), \( \rho \) and \( P \) tend to 0 but \( 2Gm/c^2r \) tends to a positive number strictly less than 1 as \( r \to r_+ - 0 \). In this sense the solution can be said to be ‘short’ if \( r_+ < +\infty \).

The equation of state for neutron stars is given by

\[
P = \frac{3}{8} Kc^3 \int_0^\infty \frac{q^4 dq}{(1 + q^2)^{1/2}}
\]

\[
\rho = 3Kc^3 \int_0^\infty (1 + q^2)^{1/2} q^2 dq
\]

\[
P = \frac{3}{8} Kc^3((2\zeta^2 + 1)(\zeta(\zeta^2 + 1)^{1/2} - \log(\zeta + (\zeta^2 + 1)^{1/2})).
\]

See [15, p. 188, (6.8.4), (6.8.5)]. In this case we have

\[
P = \frac{1}{5} K^{-2/3} \rho^{5/3}(1 + [K^{-2/3} \rho^{2/3} / c^2]_1),
\]

where \([X]_1\) stands for a convergent power series of the form \( \sum_{j \geq 1} a_j X^j \). Keeping in mind this case, we suppose the following assumption of the behavior of \( P(\rho) \) as \( \rho \to 0 \):

(A1) There are positive constants \( A, \gamma \) such that

\[
P = A\rho^\gamma (1 + [\rho^{\gamma-1}]_1)
\]

as \( \rho \to +0 \), and \( 1 < \gamma < 2 \).

Under the assumptions (A0)(A1) we can introduce the new variable \( u \) by

\[
u = \int_0^P \frac{dP}{\rho + P/c^2}, \quad (3.2)
\]

which satisfies

\[
u = \frac{A\gamma}{\gamma - 1} \rho^{\gamma-1}(1 + [\rho^{\gamma-1}]_1)
\]

as \( \rho \to +0 \). Let \((m(r), P(r)), 0 < r < r_+\), be an equilibrium, where \((0, r_+)\) is the maximal interval of existence. Then the corresponding \( v = u(r) \) satisfies

\[
r \frac{du}{dr} = - \frac{G(m + 4\pi r^3 P/c^2)}{r(1 - 2Gm/c^2r)}. \quad (3.3)
\]

Then \( u(r) \) is monotone decreasing, and moreover we have
Proposition 2 \( u(r) \to 0 \) as \( r \to r_+ - 0 \).

Proof is the same as that of [7, Lemma]. (We do not use the assumption \( \gamma > 4/3 \).)

Let us introduce the variables
\[
x = \frac{m}{ur}; \quad y = 4\pi r^2 \frac{\rho^2}{P}.
\]
The equations read
\[
\begin{align*}
r \frac{dx}{dr} &= \alpha(u) - x + x^2 \tilde{G}, \\
r \frac{dy}{dr} &= y(2 - \beta(u)x \tilde{G}), \\
r \frac{du}{dr} &= -ux \tilde{G},
\end{align*}
\]
where
\[
\begin{align*}
\alpha &= \frac{P}{u\rho} = \frac{\gamma - 1}{\gamma} + [u]_1, \\
\beta &= \left(2 \frac{dP}{dp} - \frac{u}{P}\right) = \frac{2 - \gamma}{\gamma - 1} + [u]_1, \\
\tilde{G} &= \frac{G(1 + 4\pi r^3 P/mc^2)}{1 - 2Gm/rc^2} = \frac{G(1 + u(y/c^2) x)}{1 - 2Gux/c^2}, \\
\omega &= \frac{\rho^2}{u\rho^2} = [u]_1.
\end{align*}
\]

Proposition 3 Let \( x(r) \) be corresponding to an equilibrium \((m(r), P(r)), 0 < r < r_+\). If there is \( r_0 \in (0, r_+) \) such that \( x(r_0) > 1/G \), then \( r_+ < +\infty \) and enjoys the estimate
\[
r_+ < r_0 \exp\left(\frac{1}{Gx(r_0) - 1}\right).
\]
A proof can be found in the last part of the proof of [7, Theorem 1].

As in [7] we can claim

Proposition 4 If \( 4/3 < \gamma < 2 \), then any equilibrium is short.

When \( 6/5 < \gamma \leq 4/3 \), it is known that, if \( A \) is small and if \( P(\rho) \) is sufficiently near to the exact \( \gamma \)-law \( P = A\rho^\gamma \), then any equilibrium is short. See [14]. Even if \( 1 < \gamma \leq 6/5 \), it is possible that there are short equilibria, since Proposition 3 guarantees existence of tails of short equilibria in any case and we can arbitrarily modify the equation of state in the higher density region. Anyway in this article we assume \((A0)(A1)\) only with \( 1 < \gamma < 2 \) and suppose that a short equilibrium
is given in front of us.

Let us observe roughly the behavior of a short equilibrium \((m(r), P(r))\) at the surface \(r = r_+\).

By Proposition 2 we have \(u \in C((0, r_+])\) with \(u(r_+) = 0\) and \(P(r), \rho(r)\) are so, too. Hence

\[
r \mapsto m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'
\]

belongs to \(C((0, r_+])\). Put

\[
m_+ = m(r_+) = \int_0^{r_+} 4\pi r'^2 \rho(r) dr.
\]

By definition we have \(1 - 2Gm/c^2r > 0\). Therefore

\[
\kappa = \lim_{r \to r_+} 1 - 2Gm/c^2r = 1 - 2Gm_+/c^2r_+
\]

is non-negative. We claim that \(\kappa > 0\). Otherwise, if \(\kappa = 0\),

\[
\frac{d}{dr}(1 - 2Gm/c^2r) = \frac{2G}{c^2} \left(4\pi r \rho - \frac{m}{r^2}\right) \to \frac{2Gm_+}{c^2r_+^2} = \frac{1}{r_+}
\]
as \(r \to r_+ - 0\) and

\[
1 - 2Gm/c^2r \sim -\frac{1}{r_+}(r_+ - r),
\]

which contradicts to \(1 - 2Gm/c^2r > 0\) for \(r < r_+\). Hence \(\kappa > 0\) and

\[
\frac{du}{dr} \to -K
\]
as \(r \to r_+ - 0\). Here

\[
K = \frac{Gm_+}{r_+^2 (1 - 2Gm_+/c^2r_+)}
\]

is a positive constant. Hence, since \(u \to 0\) as \(r \to r_+\), we see

\[
u \sim K(r_+ - r)
\]

and thus we have

**Proposition 5** Let \((m(r), P(r)), 0 < r < r_+,\) be a short equilibrium. Then we have

\[
\rho \sim \left(\frac{(\gamma - 1)K}{A\gamma}\right)^\frac{1}{\gamma - 1} (r_+ - r)^\frac{1}{\gamma - 1}
\]
as \(r \to r_+ - 0\), where \(K\) is the positive constant given by (3.10).
Remark. If \((m(r), P(r)), 0 < r < r_+,\) is a short equilibrium, then, on \(r \geq r_+,\) we put \(\rho = P = 0\) (vacuum), and
\[
d s^2 = \left(1 - \frac{2Gm_+}{c^2r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2Gm_+}{c^2r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]
which is the Schwarzschild’s metric. See [5, p.301]. Here we must take
\[
F(0) = \frac{1}{2} \log \frac{\rho}{\rho} = \frac{1}{2} \log \left(1 - \frac{2Gm_+}{c^2r_+}\right).
\]
Then the components of the metric are continuously differentiable across \(r = r_+\).

More precise behavior of the equilibrium at the surface can be given as follows.

Proposition 6 Assume \((A0)(A1),\) and let \((m(r), P(r)), 0 < r < r_+,\) be a short equilibrium. If \(\gamma \geq 1\) is an integer, then \(u(r)\) is analytic at \(r = r_+\).

Proof. We consider the variables
\[
X = \frac{1}{x} = \frac{ur}{m}, \quad Y = \frac{y}{x^2} = \frac{4\pi r^4u^2\rho^2}{m^2P}.
\]
Since \(du/dr < 0,\) we can take \(u\) as the independent variable instead of \(r,\) and the equations turn out to be
\[
u \frac{dX}{du} = \left(1 + \frac{1}{G}(-X + \alpha Y)\right)X, \quad (3.11a)
\]
\[
u \frac{dY}{du} = \left(2 + \beta + \frac{1}{G}(-4X + 2\alpha Y)\right)Y, \quad (3.11b)
\]
where we note
\[
\tilde{G} = G\left(1 + \frac{\omega u Y}{c^2 X}\right)/\left(1 - \frac{2Gu}{c^2 X}\right) \quad (3.12)
\]
Note that \(\tilde{G} > 0\) and \(\tilde{G} \to G/\kappa\) as \(u \to +0,\) where \(\kappa\) is the positive constant given in (3.9). Put
\[
\tilde{X} = X, \quad \tilde{Y} = \frac{Y}{\omega u}, \quad \tilde{u} = \frac{u}{\omega u} = \frac{4\pi r^4u^{\gamma-2}\rho^2}{m^2P}. \quad (3.13)
\]
We know that \(u \to \tilde{X}\) and \(u \to \tilde{Y}\) belong to \(C([0, u_\omega))\) and \(\tilde{X}|_{u=0}, \tilde{Y}|_{u=0}\) are positive. Therefore \(u \to \tilde{G} = G\left(1 + \frac{\omega u^{1/(\gamma-1)}Y}{c^2 X}\right)/\left(1 - \frac{2G}{c^2 X}\right)\) belongs to \(C([0, u_\omega)).\)

Integrating (3.11a), we have
\[
X = C_1 u \exp \left[\int_0^u \frac{1}{G}(-\tilde{X} + \alpha u^{\frac{\gamma}{2}}\tilde{Y})du\right].
\]
Since the integrand is continuous, we see $u \mapsto \bar{X}$ belongs to $C^1([0, u_0])$. Integrating (7.5a), we have

\[ Y = C_2 u \frac{1}{G} \exp \left[ \int_0^u \left( \frac{1}{G} (-4\bar{X} + 2\alpha u \bar{r} \bar{Y}) + \Omega(u) \right) du \right], \]

where

\[ 2 + \beta = \frac{\gamma}{\gamma - 1} + \Omega(u)u, \quad \Omega(u) = |u|_0. \]

Fixing $u_0 > 0$ small, we put $\bar{X}_0 := \bar{X}(u_0), \bar{Y}_0 := \bar{Y}(u_0)$. Since we know that

\[ \bar{X}(u) \to \bar{X}_* := \frac{r_+}{m_+}, \quad \bar{Y}(u) \to \bar{Y}_* := 4\pi \left[ \frac{\gamma - 1}{A_\gamma} \right] \frac{r_+^4}{m_+} \]

as $u \to 0$, we see that, if $u_0$ is sufficiently small, then $\bar{X}_0, \bar{Y}_0$ is arbitrarily near to $\bar{X}_*, \bar{Y}_*$. Now $(\bar{X}(u), \bar{Y}(u))$ is the unique solution of the integral equation

\begin{align*}
\bar{X}(u) &= \bar{X}_0 \exp \left[ - \int_u^{u_0} \frac{1}{G} (-\bar{X} + \alpha u \bar{r} \bar{Y}) du \right], 
\tag{3.14a} \\
\bar{Y}(u) &= \bar{Y}_0 \exp \left[ - \int_u^{u_0} \left( \frac{1}{G} (-4\bar{X} + 2\alpha u \bar{r} \bar{Y}) + \Omega(u) \right) du \right]. 
\tag{3.14b}
\end{align*}

Let us denote by $D_\delta$ the set $\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - \bar{X}_*| < \delta, |z_2 - \bar{Y}_*| < \delta \}$, $\delta$ being small positive number. Note that, if $|u| \leq \varepsilon_0, \varepsilon_0$ being a fixed small positive number, and if $(\bar{X}, \bar{Y}) \in D_\delta, 0 < \delta \leq \delta_0$, then $|1/G| \leq M_0, M_0$ depending upon only $\varepsilon_0, \delta_0$. In fact, since $\bar{X}_* > 0$ and $\delta$ is very small, we can suppose that $(\bar{X}, \bar{Y}) \in D_\delta$ guarantees $|\bar{X}| \geq \delta$. Let us consider the functional family $\mathcal{F}(\varepsilon, \delta)$ of all analytic functions $(\phi_1(u), \phi_2(u))$ defined and analytic for $|u| \leq \varepsilon$ and valued in $D_\delta$. The right-hand side of (3.14a), (3.14b), in which $\bar{X} = \phi_1(u), \bar{Y} = \phi_2(u)$, will be denoted by $\phi_1(u), \phi_2(u)$. Then it is easy to see that, if $|u_0| \leq \varepsilon, \varepsilon$ being sufficiently small, then $(\bar{X}_0, \bar{Y}_0) \in D_{\delta/2}$, and if $(\phi_1, \phi_2) \in \mathcal{F}(\varepsilon, \delta)$, then $(\phi_1, \phi_2) \in \mathcal{F}(\varepsilon, \delta)$. Applying the well-known fixed point theorem (see, e.g., [4, Chapter I, Théorème 7]), we have a fixed function in $\mathcal{F}(\varepsilon, \delta)$. This is our $(\bar{X}(u), \bar{Y}(u))$ by dint of the uniqueness.

Integrating

\[ \frac{1}{r} \frac{dr}{du} = -\frac{\bar{X}}{G}, \]

we see $u \mapsto r$ is analytic and $dr/du < 0$ including $u = 0$. Hence the inverse function $r \mapsto u$ is analytic at $r = r_+$. \hfill $\square$

Hereafter we suppose

(A2) \hspace{1cm} 1 < \gamma < 2 \text{ and } \frac{\gamma}{\gamma - 1} \text{ is an integer.}

Under this assumption (A2), \( \frac{1}{\gamma - 1} \) is an integer, and, since

\[ \rho = \left( \frac{\gamma - 1}{A_\gamma} \right)^{\frac{1}{\gamma - 1}} (1 + |u|_1), \tag{3.15} \]

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the density distribution \( \rho \) of the equilibrium is analytic at \( r = r_+ \), too:

\[
\rho = \left( \frac{(\gamma - 1)K}{A\gamma} \right)^{1/\gamma} (r_+ - r)^{-1/\gamma} (1 + [r_+ - r]) \tag{3.16}
\]

**Supplementary Remark 2.** Here we are considering a short equilibrium with surface \( r = r_+ \) and the Schwartschild’s metric in the exterior vacuum region. In other words, the metric

\[
ds^2 = g_{00}c^2 dt^2 + g_{11} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

is given by

\[
g_{00} = \begin{cases} 
  e^{2F} = \kappa e^{u/c^2} & (0 \leq r \leq r_+) \\
  1 - \frac{2GM_+}{c^2 r} & (r_+ < r),
\end{cases}
\]

\[-g_{11} = \begin{cases} 
  e^{2H} = \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} & (0 \leq r \leq r_+) \\
  \left( 1 - \frac{2GM_+}{c^2 r} \right)^{-1} & (r_+ < r).
\end{cases}
\]

Let us see that the components \( g_{00}, g_{11} \) are of class \( C^2 \) across \( r = r_+ \).

It is clear that \( g_{00} \) and \( g_{11} \) are continuous since \( u \to 0, m \to m_+ \) as \( r \to r_+ - 0 \) and \( \kappa = 1 - 2GM_+/c^2 r_+ \). Moreover we have

\[
\frac{d}{dr}g_{00}\bigg|_{r=r_+} = -\frac{2\kappa}{c^2} \left. \frac{du}{dr} \right|_{r=r_+} = \frac{2GM_+}{c^2 r_+},
\]

since \( du/dr \to -K \) with \( K \) given by (3.10), and

\[
\frac{d^2}{dr^2}g_{00}\bigg|_{r=r_+} = \frac{4\kappa}{c^4} \left( \frac{du}{dr} \right)^2 \bigg|_{r=r_+} = \frac{2\kappa}{c^2} \left. \frac{d^2u}{dr^2} \right|_{r=r_+} = -\frac{4GM_+}{c^2 r_+^3}.
\]

This can be verified by differentiating the equation (3.3) and seeing

\[
\frac{d^2u}{dr^2}\bigg|_{r=r_+} = \frac{2GM_+}{r_+^2 \kappa} + \frac{2}{c^2} \left( \frac{GM_+}{r_+^2 \kappa} \right)^2.
\]

Hence \( g_{00} \) is twice continuously differentiable at \( r = r_+ \). On the other hand, it is easy to see that the patched function

\[
\tilde{m}(r) = \begin{cases} 
  m(r) & (0 \leq r \leq r_+) \\
  m_+ & (r_+ < r)
\end{cases}
\]

is of class \( C^k \) iff \( \gamma < k/(k - 1) \) since

\[
\frac{dm}{dr} = 4\pi \rho r^2 = 4\pi r_+^2 \left( \frac{(\gamma - 1)K}{A\gamma} \right)^{1/(\gamma - 1)} (r_+ - r)^{-1/(\gamma - 1)} (1 + [r_+ - r]),
\]

Hence \( \tilde{m} \) and \( g_{11} \) are of class \( C^2 \) since \( \gamma < 2 \).
4 Equations for perturbations

Let us fix a short equilibrium $\rho(r)$ which is positive on $0 \leq r < r^+$. Put $m_+ = m(r^+)$. Then we can take $m$ as an independent variable and get an equilibrium $\rho = \bar{\rho}(m)$ and $r = r(m), 0 \leq m \leq m_+$. We have to consider solutions of (Ec) near to this equilibrium of the form

$$R = r(m)(1 + y), \quad (4.1)$$
$$V = r(m) v. \quad (4.2)$$

Here $y$ and $v$ are small perturbations. The equations turn out to be

$$e^{-F} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right) v, \quad (4.3)$$
$$e^{-F} \frac{\partial v}{\partial t} = \frac{4\pi}{c^2} r^2 (1 + y)^2 P r \frac{\partial}{\partial m} (rv) + G \frac{r^3 (1 + y)^3}{r^3} + \left(1 + \frac{v^2 r^2}{c^2} - \frac{2Gm}{c^2 r (1 + y)} \right) \left(1 + \frac{P}{c^2 \rho}\right)^{-1} \cdot 4\pi r (1 + y)^2 \frac{\partial P}{\partial m}. \quad (4.4)$$

Instead of $m$, let us take $r = r(m)$ as the independent variable. Since

$$\frac{dm}{dr} = 4\pi \bar{\rho} r^2,$$

we see

$$\frac{\partial}{\partial m} = \frac{1}{4\pi \bar{\rho} r^2} \frac{\partial}{\partial r}.$$

Therefore (2.24) and (4.1) imply

$$\rho = \bar{\rho} (1 + y)^{-2} \left(1 + y + r \frac{\partial y}{\partial r}\right)^{-1} \quad (4.5)$$

so that

$$\rho = \bar{\rho} \left(1 - 3y - r \frac{\partial y}{\partial r} + \left[y, \frac{\partial y}{\partial r}\right]_1\right).$$

Here $[X_1, X_2]_2$ denotes a convergent double power series of the form

$$\sum_{k_1 + k_2 \geq 2} a_{k_1 k_2} X_1^{k_1} X_2^{k_2}.$$

Let us recall that $(\bar{\rho})^{\gamma - 1} \in C^\infty([0, r^+]),$ provided that $\gamma/(\gamma - 1)$ is an integer, say, (A2).

The equation (4.4) reads

$$e^{-F} \frac{\partial v}{\partial t} = \frac{1}{c^2} (1 + y)^2 \frac{P}{\bar{\rho}} r \frac{\partial}{\partial r} (rv) + G \frac{r^3 (1 + y)^3}{r^3} + \left(1 + \frac{v^2 r^2}{c^2} - \frac{2Gm}{c^2 r (1 + y)} \right) \left(1 + \frac{P}{c^2 \rho}\right)^{-1} (1 + y)^2 \frac{\partial P}{\partial m}. \quad (4.6)$$
We have to solve (4.3)(4.6) for unknown functions \((t, r) \mapsto y, v\), where \(r\) is confined to the fixed interval \([0, r_+]\). Here \(m = m(r)\) is determined by the equilibrium through
\[
m = 4\pi \int_0^r \bar{\rho}(r)r^2 dr,
\]
and \(\rho, P(\rho), u(\rho)\) are given functions of \(\bar{\rho}(r)\) and the unknowns \(y, r\partial y/\partial r\) through (4.5).

The perturbation of \(\rho\) is expressed by (4.5). Similar expressions of \(P\) and \(u\) are necessary. If \(P(\rho)\) was the exact \(\gamma\)-law, say, if \(P = A\rho^\gamma\), then we would have
\[
P = P(1 + y)^{-2\gamma}(1 + y + r \partial y/\partial r)^{-\gamma} = \bar{P}
\]
However this exact \(\gamma\)-law is not treated by this article, since it violates the condition \(dP/d\rho < c^2\) for large \(\rho\). Our case should be treated more carefully.

We should introduce the quantity
\[
\gamma^P := \frac{\rho dP}{P d\rho}.
\]
Then under the assumption (A1) we see
\[
\gamma^P = \gamma + [u]_1
\]
and, using this function, we can express
\[
P = \bar{P} \left( 1 - \gamma^P (\bar{u})(3y + r \partial y/\partial r) - \Phi^P (\bar{u}, y, r \partial y/\partial r) \right),
\]
where
\[
\Phi^P (u, y, ry') = [u; y, ry']_{0,2}.
\]
Here \([X_0; X_1, X_2]_{0,2}\) denotes a convergent triple power series of the form
\[
\sum_{k_0 \geq 0, k_1 + k_2 \geq 2} a_{k_0, k_1, k_2} X_0^{k_0} X_1^{k_1} X_2^{k_2}.
\]
We note that
\[
\left( 1 + \frac{P}{c^2 \bar{\rho}} \right)^{-1} = \left( 1 + \frac{\bar{P}}{c^2 \bar{\rho}} \right)^{-1} \left( 1 + \frac{\bar{P}}{c^2 \bar{\rho}} \right)^{-1} (\gamma^P - 1) (3y + r \partial y/\partial r) + \left[ \bar{u}; y, r \partial y/\partial r \right]_{0,2}.
\]
Supplementary Remark 3. The letter \(r\) is used, on the one hand, as one of the co-moving coordinates for the metric (2.5), and, on the other hand, as one
of the independent variables of the equation (4.6). However these two quantities
denoted by the same letter ‘r’ do not coincide if we consider moving solutions.
Therefore, in order to clarify the relation between these two quantities, we shall
denote by r* the latter r, say one of the independent variables for (4.6).

In other words, the definition of r* = φ(t, r) is as following: Put

\[ m = f_1(r) := 4\pi \int_0^r \rho(r') r'^2 dr' \quad (0 \leq r \leq r_+) \]

along the equilibrium fixed. Then we have the inverse function \( r = f_1^{-1}(m) \)
defined on 0 \leq m \leq m_+. But along the moving solutions m is one of the
variables of the equations (2.25)(2.26) or (E_c) defined by (2.11). So we denote

\[ m = f_2(t, r) := 4\pi \int_0^r \rho(t, r') R(t, r')^2 \partial_t R(t, r') \, dr' \]

along the moving solutions under consideration. Then we put

\[ r* = \varphi(t, r) := f_1^{-1}(f_2(t, r)). \]

Let us determine the function \( \varphi(t, x) \). The function \( m = f_2(t, r) \) should satisfy

(2.20d), that is,

\[ e^{-F} \frac{\partial m}{\partial t} = -\frac{4\pi}{c^2} \text{R} \text{P} \cdot \text{V}. \]  

(4.10)

The left-hand side of (4.10) is

\[ \frac{1}{\sqrt{\epsilon}} e^{u/c^2} D f_1(r^*) \frac{\partial r^*}{\partial t} = \frac{1}{\sqrt{\epsilon}} e^{u/c^2} \cdot 4\pi \tilde{\rho}(r^*) \, (r^*)^2 \frac{\partial r^*}{\partial t}. \]

On the other hand, we are going to construct moving solutions of the form

\[ R = r^*(1 + y(t, r^*)), \quad V = r^* v(t, r^*), \]

\[ P = P(\rho), \quad \rho = \tilde{\rho}(r^*) (1 + y(t, r^*))^{-2} \left( 1 + y + r^* \frac{\partial y}{\partial r^*} \right)^{-1} \]

with \( y, v \in C^\infty([0, T] \times [0, r_+]) \), which are very small. Suppose that we have
constructed such solutions. Then the function \( \varphi(t, r) \) should satisfy

\[ \frac{\partial}{\partial t} \varphi(t, r) = -\frac{\sqrt{\epsilon}}{c^2} e^{-u/c^2} (1 + y)^2 P \frac{\partial}{\rho} v \cdot \varphi(t, r), \]  

(4.11)

where \( u = u(\rho), y, P = P(\rho), \tilde{\rho}, v \) in the right-hand side are evaluated at \((t, r^*) = (t, \varphi(t, r))\). The formula (4.11) can be considered as an ordinary differential
equation for \( \varphi(., r) \) for each fixed r, which determines \( \varphi(., r) \) provided that the
initial value \( \varphi(0, r) = f_1^{-1}(f_2(0, r)) \) is given.

But we can assume that \( \varphi(0, r) = r \) without loss of generality. In fact, \( \varphi(0, r) = r \)
means \( f_1(r) = f_2(0, r) \) and, even if \( f_1 \neq f_2(0, .) \), we can find the
change of variable \( r = \psi(r^*) \) such that \( f_1(r^*) = f_2(0, \psi(r^*)). \) Considering \( r^* \)
instead of \( r \), we can assume \( f_1(r) = f_2(0, r) \) or \( \varphi(0, r) = r \). Clearly the \( C^\infty \)-solution \( \varphi \) is uniquely determined and \( \varphi(t, r) - r \) is very small with its derivatives. Of course \( \varphi(t, 0) = 0 \) and \( \varphi(t, r_+) = r_+ \) since \( P/\dot{\rho} \) vanishes at \( r = r_+ - 0 \). Hence we have the solutions

\[ R = \varphi(t, r)(1 + y(t, \varphi(t, r))), \quad V = \varphi(t, r)v(t, \varphi(t, r)) \]

and so on as functions of the original co-moving coordinates \( t, r \).

5 Analysis of the linearized equation

We are going to analyze the linearized equations to (4.3)(4.6) and establish the existence of time periodic solutions to the linearized equations of the form

\[ y = \text{Const.} \sin(\sqrt{\lambda}t + \text{Const.}) \tilde{\psi}(r), \]

where \( \lambda > 0 \) and \( \tilde{\psi}(r) \) is an analytic function of \( r \) in a neighborhood of \([0, r_+]\).

Using the formulas listed in the last part of the preceding section, we see that the linearizations of the equations (4.3)(4.6) turn out to be

\[ e^{-F} \frac{\partial y}{\partial t} = \left( 1 + \frac{P}{c^2 \rho} \right) y \]  

\[ e^{-F} \frac{\partial v}{\partial t} = E_2 y'' + E_1 y' + E_0 y, \]

where \( y'' = \partial^2 y / \partial r^2 \), \( y' = \partial y / \partial r \) and

\[ E_2 = e^{-2H} (\rho + P/c^2)^{-1} \gamma^P P, \]

\[ E_1 = \frac{4\pi G}{c^2} e^{2H} (\rho + P/c^2) r - (\rho + P/c^2)^{-1} \left( 1 - \frac{1}{\gamma_P} \right) \frac{P'}{c^2} + \frac{3}{r} + \frac{(\gamma^P Pr)''}{\gamma^P Pr}, \]

\[ E_0 = \frac{4\pi G}{c^2} \cdot 3(\gamma^P - 1)P + \]

\[ + \left( -1 - 3\gamma^P e^{-2H} + 3(\gamma^P - 1)e^{-2H} (1 + P/c^2)^{-1} \right) \frac{P'}{r} + \]

\[ + 3e^{-2H} (\rho + P/c^2)^{-1} \frac{(\gamma^P P)'}{r}. \]

Here \( \rho, P, \gamma^P, F, H \) are abbreviations for the quantities \( \dot{\rho}(r), \dot{P} = P(\rho(r)), \gamma^F(\dot{u}(r)), \)

\( \tilde{F} = F(\dot{u}(r)) = -\frac{1}{c^2} \dot{u}(r) + \frac{1}{2} \log \kappa, \tilde{H} = -\frac{1}{2} \log \left( 1 - \frac{2Gm}{c^2 r} \right) \) along the considered equilibrium. Throughout the above manipulations we have used the equation

\[ \frac{4\pi G}{c^2} e^{2H} (\rho + P/c^2) r = F' + H', \]
which can be derived from the differentiation of

\[ e^{-2H} = 1 - \frac{2Gm}{c^2r} \]

and (3.1b), and also the relation

\[ \frac{(1 + P/c^2\rho)^{1/4}}{1 + P/c^2\rho} = \frac{1}{\rho + P/c^2} \left( 1 - \frac{1}{\gamma^2} \right) \frac{P'}{c^2}. \]

In other words, the linearized second order single equation is:

\[ \frac{\partial^2 y}{\partial t^2} + Ly = 0, \quad (5.3) \]

where

\[ Ly = \frac{a}{b} y'' - \frac{a'}{b} y' + Qy = \frac{1}{b} (ay')' + Qy, \quad (5.4) \]

\[ a = \exp \left[ \int r E_1 E_2 dr \right] = \frac{\gamma P r^4}{1 + P/c^2} e^{F + H}, \quad (5.5a) \]

\[ b = (1 + P/c^2)^{-1} P^4 e^{F + 3H}, \quad (5.5b) \]

\[ Q = -e^{2F} (1 + P/c^2) E_0. \quad (5.5c) \]

In order to investigate the spectral property of the second order linear differential operator \( L \), we reduce the eigenvalue problem

\[ Ly = \lambda y \quad (5.6) \]

to the normal form

\[ \frac{d^2 \eta}{dx^2} + q(x) \eta = \lambda \eta \quad (5.7) \]

by the Liouville transformation

\[ \xi = \int_0^r \sqrt{\frac{b}{a}} dr = \int_0^r \sqrt{\frac{\rho}{\gamma P}} e^{-F + H} dr, \quad (5.8a) \]

\[ \eta = (ab)^{1/4} y = (\gamma P \rho P)^{1/4} (1 + P/c^2)^{-1/2} e^{H} y, \quad (5.8b) \]

when the result is

\[ q = Q + \frac{a}{4b} \left( \left( \frac{a'}{a} + \frac{b'}{b} \right)' - \frac{1}{4} \left( \frac{a'}{a} + \frac{b'}{b} \right)^2 + \frac{a'}{a} \left( \frac{a'}{a} + \frac{b'}{b} \right) \right). \quad (5.9) \]

See [1, p. 275, Theorem 6].

Since

\[ \sqrt{\frac{\rho}{\gamma P}} \sim \text{Const.}(r_+ - r)^{-1/2}, \]

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we can define the finite value

\[ \xi_+ := \int_0^{r_+} \sqrt{\frac{\rho}{\gamma P F}} e^{-F+H} \, dr. \]  

(5.10)

The interval \((0, r_+)\) is mapped onto \((0, \xi_+)\).

First let us observe the behavior of \(q\) as \(\xi \to 0 (r \to 0)\). We see that \(Q = O(1)\),

\[ \frac{a'}{a} \sim \frac{4}{r}, \quad \frac{a'}{b} + \frac{b'}{a} \sim \frac{8}{r}, \quad \left(\frac{a'}{a} + \frac{b'}{b}\right)' \sim -\frac{8}{r^2}, \]

therefore

\[ q \sim 2\gamma P P\rho^{-1} e^{2F-2H} \bigg|_{r=0} \frac{1}{r^2}. \]

On the other hand we have

\[ \xi \sim \left(\gamma P P\rho^{-1} e^{2F-2H} \bigg|_{r=0}\right)^{-1/2} r. \]

Hence we have

\[ q \sim \frac{2}{\xi^2}. \]

Note that \(2 > 3/4\).

Next we observe the behavior of \(q\) as \(\xi \to \xi_+(r \to r_+)\). Note that \(P'/\rho \to -K\), where \(K\) is the constant defined by (3.10). Therefore we see that \(Q = O(1)\). Moreover we have

\[ \frac{\rho}{\gamma P} \frac{d}{d\rho} \gamma P = O(u) \to 0, \]

so that \((\gamma P)' / \gamma P = o(\rho' / \rho)\). Hence we see that

\[ \frac{a'}{a} \sim -\frac{\gamma}{\gamma - 1} \frac{1}{r_+ - r}, \quad \frac{a'}{b} + \frac{b'}{a} \sim -\frac{\gamma + 1}{\gamma - 1} \frac{1}{r_+ - r}, \]

\[ \left(\frac{a'}{a} + \frac{b'}{b}\right)' \sim -\frac{\gamma + 1}{\gamma - 1} \frac{1}{(r_+ - r)^2}. \]

Therefore we have

\[ q \sim Ke^{2F-2H} \bigg|_{r=r_+} \frac{(\gamma + 1)(3 - \gamma)}{16(\gamma - 1)} \frac{1}{r_+ - r}. \]

On the other hand we have

\[ \xi_+ - \xi \sim \frac{2}{\sqrt{(\gamma - 1)K}} e^{-F+H} \bigg|_{r=r_+} \sqrt{r_+ - r}. \]

Hence we have

\[ q \sim \frac{(\gamma + 1)(3 - \gamma)}{4(\gamma - 1)^2} \frac{1}{(\xi_+ - \xi)^2}. \]
It follows from $1 < \gamma < 2$ that

$$\frac{(\gamma + 1)(3 - \gamma)}{4(\gamma - 1)^2} > \frac{3}{4}.$$  

Therefore the both boundary points $\xi = 0, \xi_+$ are of limit point type, and [13, p. 159, Theorem X.10] gives the following conclusion, which is the same as [9, Proposition 1]:

**Proposition 7** The operator $\mathcal{T}_0, D(\mathcal{T}_0) = C_0^\infty(0, \xi_+), \mathcal{T}_0\eta = -\eta\xi_+ + \eta_+$, in $L^2(0, \xi_+)$ has the Friedrichs extension $\mathcal{T}$, a self-adjoint operator, whose spectrum consists of simple eigenvalues $\lambda_1 < \cdots < \lambda_n < \cdots \to +\infty$. In other words, the operator $\mathcal{S}_0, D(\mathcal{S}_0) = C_0^\infty(0, r_+), \mathcal{S}_0y = Ly$ in $L^2((0, r_+), bdr)$ has the Friedrichs extension $\mathcal{S}$, a self-adjoint operator with eigenvalues $(\lambda_n)_n$.

In order to investigate the structure of the linear operator $L$, we introduce the new independent variable $x$ instead of $r$ defined by

$$x := \frac{\tan^2 \theta}{1 + \tan^2 \theta} \quad \text{with} \quad \theta := \frac{\pi \xi}{2\xi_+} = \frac{\pi}{2\xi_+} \int_0^r \sqrt{\frac{\rho}{\gamma P} e^{-F+H}} dr.$$  

The interval $[0, r_+]$ of the variable $r$ is mapped onto $[0, 1]$ of $x$, and we have

$$\frac{d}{dr} = \frac{\pi}{\xi_+} \sqrt{x(1-x)} \sqrt{\frac{b}{a}} \frac{d}{dx},$$  

$$\frac{d^2}{dr^2} = \left( \frac{\pi}{\xi_+} \right)^2 \frac{b}{a} \sqrt{x(1-x)} \frac{d^2}{dx^2} + \left( \frac{1 - 2x}{2} + \frac{\xi_+}{\pi} \sqrt{x(1-x)} \sqrt{\frac{a}{b} \frac{1}{2b} \frac{d}{dx}} \right) \frac{d}{dx}.$$  

We note

$$\frac{r}{d}{dr} = x \frac{d}{dx},$$

where and hereafter $[x]$ denotes an analytic function of $x$ in a neighborhood of the interval $[0, 1]$. In fact, (5.12a) implies the following observations: as $r \to 0 (x \to 0)$, we see

$$r = \frac{\xi_+}{\pi} C_0 \sqrt{x(1 + [x]_1)} \quad \text{with} \quad C_0 = 2 \sqrt{\frac{\gamma P}{\rho} e^{F+H}} \bigg|_{r=0},$$  

and

$$\sqrt{\frac{b}{a}} = \frac{2\xi_+ \sqrt{x}}{r} (1 + [x]_1),$$

so that

$$r \frac{d}{dr} = 2x(1 + [x]_1) \frac{d}{dx};$$

as $r \to r_+(x \to 1)$, we see

$$1 - x = \left( \frac{\pi}{\xi_+} \right)^2 C_1 (r_+ - r)(1 + [r_+ - r]_1) \quad \text{with} \quad C_1 = \frac{1}{(\gamma - 1)\kappa^2 K},$$  

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(see (3.16) and note $e^{F-H} = \kappa + [r_+ - r]_1$ with $\kappa = 1 - 2Gm_+/c^2r_+$) and

$$\sqrt{\frac{b}{a}} = \frac{\pi}{\xi_+ \sqrt{1-x}}(1 + [1 - x]_1),$$

so that

$$r \frac{d}{dr} = \left(\frac{\pi}{\xi_+}\right)^2 C_1 r_+ (1 + [1 - x]_1) \frac{d}{dx}. $$

Now we can write

$$\left(\frac{\xi_+}{\pi}\right)^2 L_y = -x(1-x) \frac{d^2}{dx^2} - B \frac{dy}{dx} + \left(\frac{\xi_+}{\pi}\right)^2 Qy, \quad (5.16)$$

where

$$B = \frac{1 - 2x}{2} + \frac{\xi_+}{\pi} \sqrt{x(1-x)} \sqrt{\frac{a}{b}} \left(\frac{1}{b} \frac{d}{dr} \left(\frac{b}{a}\right) + \frac{1}{a} \frac{da}{dr}\right). \quad (5.17)$$

- As $r \to 0(x \to 0)$, we see

$$B = \frac{5}{2} + [x]_1.$$  

For

$$\frac{1}{2} \frac{a}{b} \frac{d}{dr} \left(\frac{b}{a}\right) + \frac{1}{a} \frac{da}{dr} = \frac{1}{2} \left(\gamma \rho P\right) \frac{\rho P}{\rho} + 2H' + \frac{4}{r} \left(1 + \frac{P/c^2}\right)' = \frac{4}{r}(1 + [r^2]_1)$$

and

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{\gamma \rho P}{\rho} e^{F-H}} = \left(\sqrt{\frac{\gamma \rho P}{\rho} e^{F-H}}\right)_{r=0} (1 + [r^2]_1) = \frac{\pi}{2 \xi_+ \sqrt{x}} (1 + [x]_1).$$

Clearly

$$Q = e^{-F} (1 + \frac{P/c^2}{\rho})E_0 = [r^2]_0 = [x]_0.$$  

- As $r \to r_+ (x \to 1)$, we see

$$B = -\frac{\gamma}{\gamma - 1} + [1 - x]_1.$$  

For

$$\frac{1}{2} \frac{a}{b} \frac{d}{dr} \left(\frac{b}{a}\right) + \frac{1}{a} \frac{da}{dr} = -\frac{1}{2} \frac{\gamma + 1}{\gamma - 1} \frac{1}{r_+ - r} (1 + [r_+ - r]_1)$$

$$= -\gamma + 1 \left(\frac{\pi}{\xi_+}\right)^2 C_1 \frac{1}{1-x} (1 + [1 - x]_1)$$

$$= 21$$

}
\[
\frac{a}{b} = \frac{1}{C_1}(r_+ - r)(1 + [r_+ - r]_1) = \frac{1}{C_1} \left( \frac{\xi_+}{\pi} \right)^2 (1 - x)(1 + [1 - x]_1).
\]

Clearly

\[ Q = [u]_0 = [r_+ - r]_0 = [1 - x]_0. \]

Summing up, we have the following conclusion, which is the same as [9, Proposition 3]:

**Proposition 8** We can write

\[
\left( \frac{\xi_+}{\pi} \right)^2 \mathcal{L}y = -x(1 - x) \frac{d^2 y}{dx^2} - \left( \frac{5}{2}(1 - x) - \frac{N}{2} \right) \frac{dy}{dx} + L_1(x) \frac{dy}{dx} + L_0(x)y, \tag{5.18}
\]

where \( L_1(x) = x(1 - x)[x]_1, L_0(x) = [x]_1 \). Here \( N \) is the parameter defined by

\[
N = \frac{2\gamma}{\gamma - 1} \quad \text{or} \quad \gamma = \frac{N}{N - 2}. \tag{5.19}
\]

The assumption (A2) reads that \( N \) is an even integer \( > 4 \). As long as we are concerned with investigation of the analytic structure of the operator \( \mathcal{L} \), we may assume that \( \xi_+ = \pi \) without loss of generality.

Anyway, we have the following

**Proposition 9** Let \( \lambda = \lambda_n \) be a positive eigenvalue and let \( \psi \) be an associated eigenfunction which belongs to \( L^2([0, 1]; x^{\frac{5}{2}}(1 - x)^{\frac{N}{2} - 1}dx) \). Then

\[
Y_1 = \sin(\sqrt{\lambda} + \Theta_0)\psi(x) \tag{5.20}
\]

is a time periodic solution of the linearized problem (5.3).

Thanks to Proposition 8, we can claim the following proposition on the analytic property of the eigenfunction same as [9, Proposition 4]:

**Proposition 10** We have

\[
\psi(x) = c_0(1 + [x]_1) \quad \text{as} \quad x \to 0 \tag{5.21a}
\]

\[
\psi(x) = c_1(1 + [1 - x]_1) \quad \text{as} \quad x \to 1. \tag{5.21b}
\]

Here \( c_0, c_1 \) are non-zero constants. Other independent solutions of \( \mathcal{L}y = \lambda y \) do not belong to \( L^2([0, 1]; (1 - x)^{\frac{N}{2} - 1}dx) \) as \( x \sim 1 \).

Therefore \( \psi(x) = [x]_1 \) and \( Y_1 \) is an analytic function of \( t \in \mathbb{C} \) and \( x \) on a neighborhood of \([0, 1]\) independent of \( t \).

Hereafter we fix such a time periodic function \( Y_1 \).
6  Rewriting of the equations (4.3)(4.6) using the linear operator $\mathcal{L}$

Let us go back to the system of equations (4.3)(4.6). In order to rewrite these equations using the linear operator $\mathcal{L}$, we shall use the following observations.

We are considering the perturbed $P$ such that

$$P = \bar{P}(1 - \gamma P(\bar{u})(3y + z) - \Phi P(\bar{u}, y, z)), \quad (6.1)$$

where $z = r\partial y / \partial r$.

Then we have

$$- \frac{1}{r \rho} \frac{\partial P}{\partial r} = - \frac{1}{r \rho} \frac{d\bar{P}}{dr} + \left(1 + \frac{1}{\gamma P} \partial_z \Phi P\right) \frac{1}{r \rho} \frac{\partial}{\partial r} (\bar{P} \gamma P(3y + z)) +$$

$$+ \frac{\bar{P}}{r \rho} \cdot [Q0] + \frac{1}{r \rho} \frac{d\bar{P}}{dr} \cdot [Q1], \quad (6.2)$$

where

$$[Q0] := 2(\gamma P + \partial_z \Phi P)(1 + y)^{-1} \frac{z^2}{r} + \frac{du}{dr} \left(\partial_y \Phi P - \frac{1}{\gamma P} \frac{d\gamma P}{du} (3y + z) \partial_z \Phi P\right), \quad (6.3a)$$

$$[Q1] := \Phi P - (3y + z) \partial_z \Phi P. \quad (6.3b)$$

Here we have used the relation

$$\left(\partial_y - 3\partial_z\right) \Phi P = 2(\gamma P + \partial_z \Phi P)(1 + y)^{-1} z. \quad (6.4)$$

Let us analyze

the right-hand side of (4.6) = $[R1] + [R2]$, \quad (6.5)

where

$$[R1] := -G(1 + y)\left(\frac{m}{r^3(1 + y)^3} + \frac{4\pi P}{c^2}\right) +$$

$$- \left(1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r(1 + y)}\right)(1 + P/c^2 \rho)^{-1} (1 + y)^2 \frac{\partial P}{\partial r}, \quad (6.6a)$$

$$[R2] := \frac{1}{c^2(1 + y)^2} \frac{P}{\rho} (v^2 + vw) \text{ with } w = r \frac{\partial v}{\partial r}. \quad (6.6b)$$

Let us put

$$[R1] = [R3] + [R4] + [R5] + [R6] + [R7],$$

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where

\[ [R3] := -G(1 + y) \left( \frac{m}{r^3(1 + y)^3} + \frac{4\pi P}{c^2} \right) \]

\[ = -\frac{Gm}{r^3} - \frac{4\pi GP}{c^2} + [R3L] + [R3Q], \]

\[ [R3L] := \frac{Gm}{r^3} \cdot 2y + \frac{4\pi G}{c^2} P \gamma_P (3y + z) + \frac{4\pi G}{c^2} \dot{\rho} y, \]

\[ [R3Q] := -\frac{Gm}{r^3} \left( \frac{1}{1 + y^2} - (1 - 2y) \right) + \frac{4\pi G}{c^2} \ddot{\rho} \Phi + \frac{4\pi G}{c^2} (P - \dot{P}) y; \]

\[ [R4] := -\left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{(1 + y)^2}{r\rho} \frac{dP}{dr} \]

\[ = -\left( 1 - \frac{2Gm}{c^2r} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{1}{r\rho} \frac{dP}{dr} + [R4L] + [R4Q], \]

\[ [R4L] := \left( 1 + P/c^2 \dot{\rho} \right)^{-1} (1 + P/c^2 \dot{\rho})^{-1} \frac{P}{c^2} \quad \frac{1}{r\rho} \frac{dP}{dr} + 1 \frac{dP}{dr}; \]

\[ [R5] := \left( 1 + \frac{1}{c^2} \frac{\partial_x \Phi_P}{c \partial_r} \right) \left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{(1 + y)^2}{r\rho} \frac{\partial R}{\partial r} \gamma_P (3y + z), \]

\[ [R6] := \left( 1 + \frac{1}{c^2} \frac{\partial_x \Phi_P}{c \partial_r} \right) \left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{(1 + y)^2}{r\rho} \frac{\partial P}{\partial r}; \]

\[ [R7] := \left( 1 + \frac{1}{c^2} \frac{\partial_x \Phi_P}{c \partial_r} \right) \left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{(1 + y)^2}{r\rho} \frac{dP}{dr}. \]

Then, using (3.1b), we have

\[ [R1] = [R3L] + [R3Q] + [R4L] + [R4Q] + [R5] + [R6] + [R7]. \]

Let us define \( G_1 \) by

\[ 1 + G_1 = \left( 1 + \frac{1}{c^2} \frac{\partial_x \Phi_P}{c \partial_r} \right) \left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right) \left( 1 + \frac{P}{c^2 \dot{\rho}} \right) \frac{(1 + y)^2}{1 - \frac{2Gm}{c^2r}}. \] (6.7)

Then

\[ [R5] = (1 + G_1) \left( 1 - \frac{2Gm}{c^2r} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{1}{r\rho} \frac{\partial R}{\partial r} \gamma_P (3y + z) \]

and, by definition,

\[-e^{-2F}(1 + P/c^2 \dot{\rho})^{-1} \mathcal{L} y = [R3L] + [R4L] + \]

\[ + \left( 1 - \frac{2Gm}{c^2r} \right) (1 + P/c^2 \dot{\rho})^{-1} \frac{1}{r\rho} \frac{\partial R}{\partial r} \gamma_P (3y + z) \]

\[ = [R3L] + [R4L] + \frac{1}{1 + G_1} [R5]. \]
This implies

\[ [R1] = -(1 + G_1)e^{-2F}(1 + \frac{P}{c^2} \frac{\varphi}{\rho})^{-1} \mathcal{L}y + \]
\[ - G_1([R3L] + [R4L]) + [R3Q] + [R4Q] + [R6] + [R7]. \]

Now, putting

\[ H_1 := e^F \mathcal{F}^2 (1 + \frac{P}{c^2} \frac{\varphi}{\rho})^{-1}(1 + G_1), \]
\[ H_2 := e^F G_2, \]
\[ G_2 := (1 + G_1)([R3L] + [R4L]) - [R3] - [R4] + [R6] - [R7] - [R2], \]

we can write

\[ e^F \times \text{(the right-hand side of (4.6))} = -H_1 \mathcal{L}y - H_2. \]

The following observation will play a crucial role in the analysis of the equation as in [9].

**Proposition 11** There is an analytic function \( \hat{a} \) of \( 1 - x, y, z, v, w, y', y'' \) such that

\[ (\partial_z H_1) \mathcal{L}y + \partial_z H_2 = (1 - x)\hat{a} \]

as \( x \to 1 \).

**Proof.** For the sake of abbreviations, hereafter we will denote

\[ Q_1 \equiv Q_0 \]

if there is an analytic function \( \Omega(1 - x, y, z, v, w, y', y'') \) such that

\[ Q_1 = Q_0 + (1 - x)\Omega. \]

We are considering

\[ (\partial_z H_1) \mathcal{L}y + \partial_z H_2 = (\partial_z F \cdot H_1 + e^{-2F}(1 + \frac{P}{c^2} \frac{\varphi}{\rho})^{-1} \partial_z G_1) \mathcal{L}y + \]
\[ + \partial_z F \cdot H_2 + e^F \partial_z G_2. \]

First we note that (2.27) and (4.5) imply

\[ \partial_z F = \frac{1}{c^2} \rho \frac{du}{dp}(1 + y + z)^{-1} \]

and that

\[ \rho \frac{du}{dp} = (\gamma - 1)u(1 + |u|_1), \quad u = \bar{u}(1 + [x; y, z]_1) \equiv 0. \]
Here \([x; y, z]\) stands for an analytic function of \(x\) in a neighborhood of \([0, 1]\) and \(y, z\) of a neighborhood of \((0, 0)\) of the form \(\sum_{k_1+k_2 \geq 1} a_{k_1k_2}(x)y^{k_1}z^{k_2}\). Therefore \(\partial_z F \equiv 0\) and

\[
(\partial_z H_1)Ly + \partial_z H_2 \equiv e^F[S],
\]

where

\[
[S] = (\partial_z G_1)e^{-2F}(1 + \frac{\dot{P}}{c^2\rho})^{-1}Ly + \partial_z G_2 =
\]

\[
= -(\partial_z G_1) \left( 1 - \frac{2Gm}{c^2r} \right) \left( 1 + \frac{\dot{P}}{c^2\rho} \right)^{-1} \frac{1}{r\rho} \frac{\partial}{\partial r} \gamma P(3y + z) +
\]

\[
+ (1 + G_1) \frac{\partial}{\partial z} \left( [R3L] + [R4L] \right) - \frac{\partial}{\partial z} \left( [R3] + [R4] + [R6] + [R7] + [R2] \right).
\]

But, keeping in mind that \(\dot{P}/\dot{\rho} \equiv P/\rho \equiv 0\) and that

\[
\frac{\partial P}{\partial z} = -\rho \frac{dP}{d\rho}(1 + y + z)^{-1} \equiv 0,
\]

\[
\frac{\partial}{\partial z} \left( \frac{P}{\rho} \right) = \left( -\frac{dP}{d\rho} + \frac{R}{\rho} \right)(1 + y + z)^{-1} \equiv 0,
\]

we see

\[
-(\partial_z G_1) \left( 1 - \frac{2Gm}{c^2r} \right) \left( 1 + \frac{\dot{P}}{c^2\rho} \right)^{-1} \frac{1}{r\rho} \frac{\partial}{\partial r} \gamma P(3y + z) \equiv
\]

\[
= -\partial_z^2 \Phi P \left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right)(1 + y)^2 \frac{1}{r\rho} \frac{dP}{dr}(3y + z).
\]

On the other hand it is easy to see

\[
\frac{\partial}{\partial z} [R3L] \equiv \frac{\partial}{\partial z} [R4L] \equiv \frac{\partial}{\partial z} [R3] \equiv \frac{\partial}{\partial z} [R4] \equiv \frac{\partial}{\partial z} [R6] \equiv \frac{\partial}{\partial z} [R2] \equiv 0
\]

and

\[
\frac{\partial}{\partial z} [R7] \equiv -\left( 1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} \right)(1 + y)^2 \frac{1}{r\rho} \frac{dP}{dr}(3y + z) \partial_z^2 \Phi P.
\]

Hence we have \([S] \equiv 0\) so that

\[
(\partial_z H_1)Ly + \partial_z H_2 \equiv 0.
\]

This was to be shown. □

**Remark.** Note that \(\partial[R7]/\partial z \neq 0\). In fact, we have

\[
\frac{1}{r\rho} \frac{dP}{dr} \rightarrow -\frac{K}{r_+} \neq 0
\]

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and
\[
\partial^2 \Phi^P = -\frac{P}{\rho} \left( \frac{d}{d\rho} \rho \frac{dP}{d\rho} + \frac{dP}{d\rho} \right) (1 + y + z)^{-2} (1 - \gamma^P (3y + z) - \Phi^P) \\
\rightarrow - (\gamma + 1)(1 + y + z)^{-2} (1 - \gamma (3y + z) - \Phi^P (0, y, z)) \neq 0
\]
as \(x \to 1\).

Now, putting
\[
J := e^F (1 + P/c^2 \rho), \tag{6.9}
\]
we rewrite the system of equations (4.3)(4.6) as
\[
\begin{align*}
\frac{\partial y}{\partial t} - Jv &= 0, \tag{6.10a} \\
\frac{\partial v}{\partial t} + H_1 Ly + H_2 &= 0. \tag{6.10b}
\end{align*}
\]
Here the unknown functions are \((t, x) \mapsto y, v\).

7 Framework to apply the Nash-Moser(-Hamilton) theorem

Having fixed a time periodic solution \(Y_1\) of the linearized equation, we put
\[
y = \varepsilon (Y_1 + Y), \tag{7.1}
\]
\[
z = r \frac{\partial y}{\partial r} = \varepsilon (Z_1 + Z) \quad \text{with} \quad Z_1 = r \frac{\partial Y_1}{\partial r}, \tag{7.2}
\]
\[
v = \varepsilon (V_1 + V) \quad \text{with} \quad V_1 = \frac{1}{J^o} \frac{\partial Y_1}{\partial t}. \tag{7.3}
\]
Here
\[
J^o := J \bigg|_{y = z = 0} = e^F (1 + P/c^2 \rho), \tag{7.4}
\]
and \(Y, Z = r \partial Y/\partial r, V\) are new unknown functions. The parameter \(\varepsilon\) will be taken sufficiently small.

Now the system of equations turns out to be
\[
\begin{align*}
\frac{\partial Y}{\partial t} &= Jv - (\Delta J)V_1 = (J - J^o)V_1 \tag{7.5a} \\
\frac{\partial V}{\partial t} &= H_1 Ly + (\Delta H_1) (LY_1) + \frac{1}{\varepsilon} \Delta H_2 = \tag{7.5b} \\
&= - \left( H_1 - \frac{1}{J^o} \right) (LY_1) - \frac{1}{\varepsilon} H_2.
\end{align*}
\]
where

\[(J - J^0)^o := (J - J^0) \bigg|_{Y=Z=0} = J \bigg|_{y=\varepsilon Y_1, z=\varepsilon Z_1} - J^0, \quad (7.6a)\]

\[\Delta J := J - J^0 - (J - J^0)^o = J - J \bigg|_{y=\varepsilon Y_1, z=\varepsilon Z_1}, \quad (7.6b)\]

\[\left( H_1 - \frac{1}{J^0} \right)^o := \left( H_1 - \frac{1}{J^0} \right) \bigg|_{Y=Z=V=0} = \]

\[= H_1 \bigg|_{y=\varepsilon Y_1, z=\varepsilon Z_1, v=\varepsilon V_1} - \frac{1}{J^0}, \quad (7.6c)\]

\[\Delta H_1 := H_1 - \frac{1}{J^0} - \left( H_1 - \frac{1}{J^0} \right)^o = \]

\[= H_1 - H_1 \bigg|_{y=\varepsilon Y_1, z=\varepsilon Z_1, v=\varepsilon V_1}, \quad (7.6d)\]

\[H_2^o := H_2 \bigg|_{y=\varepsilon Y_1, z=\varepsilon Z_1, v=\varepsilon V_1} \quad (7.6e)\]

\[\Delta H_2 = H_2 - H_2^o. \quad (7.6f)\]

Let us introduce the vector-valued unknown function

\[\vec{w} = \begin{bmatrix} Y \\ V \end{bmatrix}, \quad (7.7)\]

We put

\[\Psi(\vec{w}) = \begin{bmatrix} \text{the left-hand side of (7.5a)} \\ \text{the left-hand side of (7.5b)} \end{bmatrix}, \quad (7.8)\]

and

\[\vec{c} = \frac{1}{\varepsilon} \begin{bmatrix} \text{the right-hand side of (7.5a)} \\ \text{the right-hand side of (7.5b)} \end{bmatrix}, \quad (7.9)\]

The equation to be solved now is

\[\Psi(\vec{w}) = \varepsilon \vec{c}. \quad (7.10)\]

We are going to apply the Nash-Moser(-Hamilton) theorem to find \(\vec{w} = \Psi^{-1}(\varepsilon \vec{c})\). To do it, we must analyze the Fréchet derivative \(D\Psi\) of the mapping \(\Psi\) at a given fixed \(\vec{w} \in C^\infty([0, T]\times [0, 1])\). Introducing the new variable

\[\vec{h} = \begin{bmatrix} h \\ k \end{bmatrix}, \quad (7.11)\]

the Fréchet derivative is defined by

\[D\Psi(\vec{w})\vec{h} = \lim_{s \to 0} \frac{1}{s} (\Psi(\vec{w} + s\vec{h}) - \Psi(\vec{w})) = \begin{bmatrix} DP1 \\ DP2 \end{bmatrix}, \quad (7.12)\]
where
\[DP1 = \frac{\partial}{\partial t} h - J \cdot k + \left( (\partial_y J) v + (\partial_z J) v r \frac{\partial}{\partial r} \right) h, \]
\[DP2 = \frac{\partial}{\partial t} k + H_1 \cdot \mathcal{L} h + \left( (\partial_y H_1) \mathcal{L} y + \partial_y H_2 + (\partial_z H_1) \mathcal{L} y + \partial_z H_2) r \frac{\partial}{\partial r} \right) h + \left( (\partial_v H_1) \mathcal{L} y + \partial_v H_2 + \partial_w H_2 \cdot r \frac{\partial}{\partial r} \right) k. \]

Thanks to Proposition 11, we can claim the following

**Proposition 12** We have
\[\left( \frac{\partial z}{\partial J} \right)_r \frac{\partial}{\partial r} = \left( [x; y, D_y, D^2_y, v, Dv]_0 \cdot x(1 - x) \frac{\partial}{\partial x}, \right. \]
\[[\partial_z H_1] \frac{\partial}{\partial r} = [x; y, D_y, D^2_y, v, Dv]_0 \cdot x(1 - x) \frac{\partial}{\partial x}, \]
\[\left. \partial_w H_2 \cdot r \frac{\partial}{\partial r} = [x; y, D_y, D^2_y, v, Dv]_0 \cdot x(1 - x) \frac{\partial}{\partial x}. \right\]

Here \( D = \partial/\partial x \).

**Proof.** Since \( r \frac{\partial}{\partial r} = 2x(1 + [x]_1) \frac{\partial}{\partial x} \)
as \( x \rightarrow 0 \) \((r \rightarrow 0)\), the problem is concentrated to the situation as \( x \rightarrow 1 \) \((r \rightarrow r_+)\). Now, since
\[\partial_z J = (\partial_z F) J + e^F \frac{1}{c^2} \frac{\partial}{\partial z} \left( \frac{P}{\rho} \right) = \frac{1}{c^2} e^F \frac{P}{\rho} (1 + y + z)^{-1}, \]
it is clear that \( \partial_z J \equiv 0 \) \((\text{mod}(1 - x))\), that is, (7.14a). (7.14b) is the result of Proposition 11. By definition we have
\[\partial_w H_2 = e^F \partial_w G_2 = -e^F \partial_w [R2] = -e^F \frac{1}{c^2} (1 + y)^2 \frac{P}{\rho} v \equiv 0, \]
that is, (7.14c). □

In the sequel we can claim that there are analytic functions \( a_{01}, a_{00}, a_{11}, a_{10}, a_{21}, a_{20} \) of \( x, y, D_y, D^2_y, v, Dv \), where \( D = \partial/\partial x, y = \varepsilon(Y_1 + Y), v = \varepsilon(V_1 + V) \), such that the components of \( D\mathcal{P}(\vec{w})\vec{h} \) can be written as
\[DP1 = \frac{\partial}{\partial t} h - J k + (a_{01} x(1 - x) D + a_{00}) h, \]
\[DP2 = \frac{\partial}{\partial t} k + H_1 \mathcal{L} h + (a_{11} x(1 - x) D + a_{10}) h + (a_{21} x(1 - x) D + a_{20}) k. \]
We note that $a_{01}, \ldots, a_{20} = O(\varepsilon)$ provided that $Y, DY, V, DV = O(1)$. On the other hand we note, by definition, that
\[
J = e^F (1 + P/c^2 \rho) = e^F (1 + \bar{P}/c^2 \bar{\rho})(1 + [x; y, Dy]_1)
\]
and
\[
H_1 = e^{F-2F} (1 + \bar{P}/c^2 \bar{\rho})^{-1}(1 + G_1) = e^{-F}(1 + \bar{P}/c^2 \bar{\rho})^{-1}(1 + [x; y, Dy]_1 + v^2[x; y, Dy]_0).
\]
Hence
\[
J = e^F (1 + P/c^2 \rho)(1 + O(\varepsilon)),
\]
\[
H_1 = e^{-F}(1 + \bar{P}/c^2 \bar{\rho})^{-1}(1 + O(\varepsilon))
\]
provided that $Y, DY = O(1)$.

**Remark.** We see $J \to 1, H_1 \to 1 + [y, z]_1$ as $1/c^2 \to 0$, while $\bar{P}/\bar{\rho}, \bar{u}$ are supposed to be bounded. (The equilibrium depends upon the central density $\rho_c$ and the speed of light $c$.) But we do not discuss the details of the non-relativistic limit in this article.

## 8 Main conclusion

Now we are ready to propose the main conclusion of this article.

**Theorem 1** Given $T > 0$, there is a positive number $\varepsilon_0(T)$ such that, for $|\varepsilon| \leq \varepsilon_0(T)$, there is a solution $\vec{w} \in C^\infty([0, T] \times [0, 1])$ of (7.10) such that
\[
\sup_{j+k \leq n} \left\| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^k \vec{w} \right\|_{L^\infty([0, T] \times [0, 1])} \leq C_n |\varepsilon|,
\]
and hence a solution $(y, v) \in C^\infty([0, T] \times [0, r_+])$ of (4.3)(4.6) of the form
\[
y = \varepsilon Y_1 + O(\varepsilon^2).
\]

Note that for this solution the component $R$ of the metric (2.5) behaves like
\[
R = r(1 + \varepsilon Y_1 + O(\varepsilon^2)),
\]
and the density distribution enjoys
\[
\rho = \begin{cases} 
C(t)(r_+ - r)^{1/(\gamma - 1)}(1 + O(r_+ - r)) & (0 \leq r < r_+) \\
0 & (r_+ \leq r)
\end{cases}
\]
Here $C(t)$ is a smooth positive function of $t$. 

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In other words, the value $R_+(t)$ of the Eulerian coordinate $R$ at the surface of the star $r = r_+$ is approximately oscillating as

$$R_+(t) = r_+(1 + \varepsilon \sin(\sqrt{\lambda} t + \Theta_0)) + O(\varepsilon^2).$$

A proof can be given by an application of the Nash-Moser(-Hamilton) theorem ([3, p.171, III.1.1.1.]) as in [9], [8]. The discussion is quite parallel. Therefore, omitting the repetitions of the details, we will explain only the points for which some modifications are necessary.

First the mapping $\Psi$ is considered on the tame spaces $\vec{E}$ and $\vec{E}_0$. Here $\vec{E} = E \times E$ with $E = C^1([0,T] \times [0,1])$ and $\vec{E}_0 = E_0 \times E_0$ with $E_0 = \{ \phi \in E \mid \phi = 0 \text{ at } t = 0 \}$. Since $\vec{E}$ admits the gradings of norms as in [9], $\vec{E}$ is a tame space as the direct Cartesian product. The domain of $\Psi$ is $\vec{U}$, the set of all functions $\vec{w} = (\vec{Y}; \vec{V})$ such that $jYj + jdYj + jVj + dVj < 1$.

We consider $\varepsilon$ such that $|\varepsilon| \leq \varepsilon_1$, $\varepsilon_1$ being a fixed sufficiently small positive number. The mapping $\Psi$ is $\vec{U}$, the set of all functions $\vec{w} = (Y, V)^T \in \vec{E}_0$ such that

$$|Y| + |DY| + |V| + |DV| < 1.$$

Introducing the operator

$$\Lambda = x(1-x) \frac{d^2}{dx^2} + \left( \frac{5}{2} (1-x) - \frac{N}{2} x \right) \frac{d}{dx}, \quad (8.1)$$

just as [9, (20)], we rewrite the second component of $D\Psi(\vec{w})\vec{h}$ as

$$[DP2] = \frac{\partial}{\partial t} k - H_1 \lambda h + b_1 \dot{D} h + b_0 h + a_{21} \dot{D} k + a_{20} k, \quad (8.2)$$

where

$$\dot{D} = x(1-x) \frac{\partial}{\partial x}, \quad (8.3a)$$

$$b_1 = \frac{H_1 L_1}{x(1-x)} + a_{11}, \quad (8.3b)$$

$$b_0 = H_1 L_0 + a_{10}. \quad (8.3c)$$

Then $b_1, b_0, a_{21}, a_{20}$ are analytic functions of $x, y, Dy, D^2 y, v, Dv$.

Let us introduce the Hilbert spaces $\vec{X} = \vec{X}_0, \vec{X}_1, \vec{X}_2$, just in the same manner as [9], by

$$\vec{X} = L^2((0,1); x^{1/2} (1-x)^{N-1} dx),$$

$$\vec{X}_1 = \{ \phi \in \vec{X} \mid \dot{D}\phi := \sqrt{x(1-x)} \frac{d\phi}{dx} \in \vec{X} \}$$

$$\vec{X}_2 = \{ \phi \in \vec{X}_1 \mid -\Lambda \phi \in \vec{X} \}.$$
We write the equation
\[ D\Psi(\vec{w})\vec{h} = \vec{g}, \]
where \( \vec{g} = (g_1, g_2)^T \) is a given function in \( \bar{E} \), as
\[ \frac{\partial}{\partial n} \begin{bmatrix} h \\ k \end{bmatrix} + \begin{bmatrix} a_1 & -J \\ A & a_2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \] (8.4)
where
\[ a_1 = a_{01} \dot{\mathcal{D}} + a_{00} \]
(8.5a)
\[ a_2 = a_{21} \dot{\mathcal{D}} + a_{20} \]
(8.5b)
\[ A = -H_1 \lambda + b_1 \dot{\mathcal{D}} + b_0. \]
(8.5c)

Then the standard calculation leads us to the equality
\[ \frac{1}{2} \frac{d}{dt} \left[ \|k\|^2 + \left( \frac{H_1}{J} \dot{\mathcal{D}} h \big| \dot{\mathcal{D}} h \right) \right] + \left( \beta_1 \dot{\mathcal{D}} h \big| \dot{\mathcal{D}} h \right) + \left( \beta_2 \dot{\mathcal{D}} h \big| h \right) + \left( \beta_3 \dot{\mathcal{D}} h \big| k \right) + \left( \beta_4 h \big| k \right) = \left( \frac{H_1}{J} \dot{\mathcal{D}} h \big| \dot{\mathcal{D}} g_1 \right) + \left( k \big| g_2 \right), \]
where
\[ \beta_1 = -\frac{1}{4} (3 + (N + 3)x + 2\dot{\mathcal{D}}) \frac{H_{1a_{00}}}{J} - \frac{1}{2} \frac{\partial}{\partial t} \frac{H_1}{J} (\dot{\mathcal{D}} a_{01} + a_{00}), \]
\[ \beta_2 = \frac{H_1}{J} \dot{\mathcal{D}} a_{00}, \]
\[ \beta_3 = -\frac{H_1}{J} \dot{\mathcal{D}} A + \dot{\mathcal{D}} H_1 + \sqrt{x(1-x)}(b_1 + a_{21}), \]
\[ \beta_4 = b_0, \]
\[ \beta_5 = a_{20}. \]

Here
\[ (\phi|\psi) = (\phi|\psi)_x = \int_0^1 \phi \overline{\psi} x^{\frac{3}{2}} (1 - x)^{\frac{3}{2} - 1} dx \]
and \( \|\phi\| = \|\phi\|_x = \sqrt{(\phi|\phi)_x} \),
and we have used the formula
\[ (\alpha \dot{\mathcal{D}} h \big| \dot{\mathcal{D}} h) = (\alpha^* \dot{\mathcal{D}} h \big| \dot{\mathcal{D}} h) \]
with \( \alpha^* = -\frac{1}{4} (3 + (N + 3)x + 2\dot{\mathcal{D}})\alpha \),
which holds for \( h \in \mathcal{X}^2 \) and \( \alpha \in C^\infty([0,1]) \), together with [9, Proposition 8].

Since \( \vec{w} \) is confined to \( \bar{\Omega} \) and \( |\vec{e}| \) is restricted \( \leq \varepsilon_0 \), we can assume
\[ \frac{1}{M_0} \leq J \leq M_0, \quad \frac{1}{M_0} \leq H_1 \leq M_0 \]
with a constant $M_0$ independent of $\vec{w}$.

Now the energy

$$E := \|k\|^2 + \left(\frac{H_1}{J} \dot{h} |\dot{h}| \right)$$

enjoys the inequality

$$\frac{1}{2} \frac{dE}{dt} \leq M \left(\|\vec{h}\|_H^2 + \|\vec{h}\|_B \|\vec{g}\|_B\right),$$

where $\vec{H} = \mathbb{X}^1 \times \mathbb{X}$ and

$$\|(\phi, \psi)^T\|_B^2 = \|\phi\|^2_{\mathbb{X}^1} + \|\psi\|^2_{\mathbb{X}} = \|\phi\|^2 + \|\dot{\phi}\|^2 + \|\psi\|^2,$$

and

$$M = \sum_{1 \leq j \leq 5} \|\beta_j\|_{L^\infty} + (M_0)^2 + 1.$$

Since

$$\frac{1}{(M_0)^2} (\|k\|^2 + \|\dot{h}\|^2) \leq E \leq (M_0)^2 (\|k\|^2 + \|\dot{h}\|^2),$$

using the same Gronwall’s argument as [9, Proposition 9], [8, Lemma 3], we see that the initial value problem for the equation (8.4) with the initial condition

$$h = k = 0 \quad \text{at} \quad t = 0$$

admits a unique solution $\vec{h} = (h, k)^T$ in $C([0, T], \mathbb{X}^2 \times \mathbb{X}^1)$ for given $\vec{g} \in C([0, T], \mathbb{X}^3 \times \mathbb{X})$, which enjoys the energy estimate

$$\|\vec{h}\|_B \leq C \int_0^t \|\vec{g}(t')\|_B dt'.$$

Therefore $D\mathfrak{P}(\vec{w})$ admits an inverse, and its tame estimates can be shown in the same manner as [9]. An outline of this procedure can be found in Appendix. This completes the proof of the main conclusion.

**9 Cauchy problems**

As a supplement let us consider the Cauchy problem associated with equations (6.10a)(6.10b), that is, (CP):

$$\begin{aligned}
\frac{\partial y}{\partial t} + Ju &= 0, \quad \frac{\partial v}{\partial t} + H_1 Ly + H_2 = 0, \quad (t \geq 0, 0 \leq x \leq 1) \\
 y|_{t=0} &= \psi_0(x), \quad v|_{t=0} = \psi_1(x).
\end{aligned} \quad (9.1)$$

Here $\psi_0$ and $\psi_1$ are functions given in $C^\infty([0, 1])$.  

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Let us recall that

\[ J = e^F (1 + P/c^2 \rho) = J(x, y, z) \]

is an analytic function of \( x \) (in a neighborhood of \([0, 1]\)), \( y \) (small) and \( z = r \frac{\partial y}{\partial r} = x [x | \frac{\partial y}{\partial x}] \) (small), where \([x x]\) stands for an analytic function of \( x \) in a neighborhood of \([0, 1]\); that \( H_1 \) and \( H_2 \) are analytic functions of \( x, y, z, v \) and \( w = r \frac{\partial v}{\partial r} \) (quadratic in \( v/c, w/c \)); and that the linear operator \( L \) has the form

\[
Ly = -x(1-x) \frac{d^2y}{dx^2} - \left( \frac{5}{2} (1-x) - \frac{N}{2} x \right) \frac{dy}{dx} + l_1(x) x(1-x) \frac{dy}{dx} + L_0(x) y,
\]

where \( l_1 \) and \( L_0 \) are analytic functions of \( x \) in a neighborhood of \([0, 1]\).

We claim the following

**Theorem 2** For any given \( T > 0 \) there exists a sufficiently small positive number \( \delta \) such that if \( \psi_0, \psi_1 \in C^\infty([0, 1]) \) satisfy

\[
\max_{k \leq K} \left\{ \left\| \left( \frac{d}{dx} \right)^k \psi_0 \right\|_{L^\infty}, \left\| \left( \frac{d}{dx} \right)^k \psi_1 \right\|_{L^\infty} \right\} \leq \delta
\]

then there exists a unique solution \((y, v)\) of \((CP)\) in \( C^\infty([0,T] \times [0,1]) \). Here \( K \) is a sufficiently large number depending only upon \( \gamma \).

Proof can be done in a almost same manner as that of Theorem 2 of [9]. Let us take the functions

\[
y_1^* = \psi_0(x) + \tau J^*(x) \psi_1(x), \quad v_1^* = \psi_1(x),
\]

which satisfy the initial conditions, where \( J^*(x) = J(x, 0, 0) \) as \((7.4)\). Then we seek a solution \((y, v)\) of the form

\[
y = y_1^* + Y, \quad v = v_1^* + V.
\]

The initial condition for \( \vec{w} := (Y, V)^T \) is

\[
\vec{w}|_{t=0} = (0, 0)^T
\]

and the equations to be satisfied by \( \vec{w} = (Y, V)^T \) are

\[
\frac{\partial Y}{\partial t} - J V - (\Delta J) v_1^* = c_1, \tag{9.6a}
\]

\[
\frac{\partial V}{\partial t} + H_1 L Y + (\Delta H_1) L y_1^* + \Delta H_2 = c_2, \tag{9.6b}
\]

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where
\[ J = J(x, y_1^*, Y, z_1^* + Z), \quad \text{with} \quad Z := r \frac{\partial Y}{\partial r}, \]  
\[ \Delta J = J(x, y_1^* + Y, z_1^* + Z) - J(x, y_1^*, z_1^*), \]  
\[ c_1 = J(x, y_1^*, z_1^*) - J(x, 0, 0), \]  
\[ H_1 = H_1(x, y_1^* + Y, z_1^* + Z, v_1^* + V, w_1^* + W), \quad \text{with} \quad W = r \frac{\partial V}{\partial r}, \]  
\[ \Delta H_1 = H_1(x, y_1^* + Y, z_1^* + Z, v_1^* + V, w_1^* + W) - H_1(x, y_1^*, z_1^*, v_1^*, w_1^*) \]  
\[ c_2 = -H_1(x, y_1^*, z_1^*, v_1^*, w_1^*) L y_1^* - H_2(x, y_1^*, z_1^*, v_1^*, w_1^*). \] 

The problem can be written as
\[ \mathfrak{P}(\bar{w}) = \bar{c}, \]  
where
\[ \mathfrak{P}(\bar{w}) = \begin{bmatrix} \text{the left-hand side of (9.6a)} \\ \text{the left-hand side of (9.6b)} \end{bmatrix} \]  
\[ \bar{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \] 

Then the Nash-Moser(-Hamilton) theorem can be applied in the same manner as the proof of Theorem 1, since the Fréchet derivative of \( \mathfrak{P} \) has the same form as (7.12)(7.13). This completes the proof.

**Remark.** The initial data read
\[ R_{|t=0} = r(1 + \psi_0(x(r))), \]  
\[ \left. \frac{\partial R}{\partial t} \right|_{t=0} = \frac{1}{c} \left( 1 - \frac{2Gm_+}{c^2 r^*} \right) \exp \left[ - \frac{1}{c^2} u(\rho^0) \right] r \psi_1(x(r)), \]  
where
\[ \rho^0 = \rho(r)(1 + \psi_0)^{-2} \left( 1 + \psi_0 + r \frac{d \psi_0}{d r} \right)^{-1}. \]

**Supplementary Remark 4.** Let us consider moving solutions constructed in Section 8 or 9, which are defined on \( 0 \leq t \leq T, 0 \leq r \leq r_+ \). We should discuss how to extend the metric onto the exterior vacuum region \( r > r_+ \). We owe the idea to [10].

If a spherically symmetric extension to the vacuum region is possible, the Birkhoff’s theorem reads that it should be the Schwarzschild’s metric
\[ ds^2 = \left( 1 - \frac{2Gm_+}{c^2 R_s^3} \right) c^2 (dt^2) - \left( 1 - \frac{2Gm_+}{c^2 R_s^3} \right)^{-1} (dR^3)^2 - (R^3)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]
Here \( t^* = t^*(t, r), R^3 = R^3(t, r) \) are smooth functions of \( 0 \leq t \leq T, r_+ \leq r < \infty. \) We have: 

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There are $t^2(t, r), R^2(t, r)$ such that the components of the metric are of class $C^1([0, T] \times [0, +\infty))$.

Let us verify this. We are considering the patched metric
\[ ds^2 = g_{00}c^2dt^2 + 2g_{01}cdtdr + g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2 \theta d\phi^2), \]
where
\[
\begin{align*}
g_{00} &= \begin{cases} 
\kappa e^{-2u/c^2} & (0 \leq r < r_+) \\
K^t(t, r) \left( \frac{\partial t}{\partial t} \right)^2 - \frac{1}{c^2} (K^t)^{-1} \left( \frac{\partial R^2}{\partial t} \right)^2 & (r_+ < r)
\end{cases} \\
g_{01} &= \begin{cases} 
0 & (0 \leq r \leq r_+) \\
cK^t(t, r) \frac{\partial t}{\partial t} \frac{\partial t}{\partial r} - \frac{1}{c} (K^t)^{-1} \frac{\partial R^2}{\partial t} \frac{\partial R^2}{\partial r} & (r_+ < r)
\end{cases} \\
g_{11} &= \begin{cases} 
-\left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2R} \right)^{-1} \left( \frac{\partial R^2}{\partial r} \right)^2 & (0 \leq r \leq r_+)
\end{cases} \\
g_{22} &= \begin{cases} 
-R^2 & (0 \leq t \leq r_+) \\
-(R^2)^2 & (r_+ < r)
\end{cases}
\end{align*}
\]

Here
\[ K^t = 1 - \frac{2Gm}{c^2R}. \]

Let us assume $R = R^2, \partial_r R = \partial_r R^2$ at $r = r_+$ in order that $g_{22}$ be of class $C^1$.

First in order that $g_{00}$ be continuous across $r = r_+$ we require
\[ \frac{\partial t^2}{\partial t} = \sqrt{\kappa} (K^t)^{-1} \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2R} \right)^{1/2}, \quad (9.11) \]
on $r = r_+$, where $V = V(t, r_+ - 0)(= \frac{1}{\sqrt{\kappa}} \frac{\partial R}{\partial t})$. In order that $g_{01}$ be continuous, we require
\[ \frac{\partial t^2}{\partial r} = \frac{1}{c} (K^t)^{-1} \left( 1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2R} \right)^{-1/2} V \frac{\partial R}{\partial r} \quad (9.12) \]
on $r = r_+$. It can be shown that (9.12) is sufficient in order that $g_{11}$ be continuous across $r = r_+$. Summing up, $g_{\mu\nu}$ are continuous if (9.11) and (9.12) hold. Note that, since
\[ K^t \doteq \kappa, \quad 1 + \frac{V^2}{c^2} = \frac{Gm}{c^2R} \doteq \kappa, \quad \frac{\partial R}{\partial r} \doteq 1, \]
the right-hand side of (9.12) $\doteq V/c^2$ so that $\partial t^2/\partial r \neq 0$ and $t^2$ should actually depend upon $r$ if $V \neq 0$, that is, if the solution is actually moving.
By a tedious calculation we can show that the differentiation of (9.11) with respect to \( t \) gives the continuity of \( \partial_t g_{00} \). On the other hand the continuity of \( \partial_r g_{01} \) reads a condition of the form

\[
K^2 \frac{\partial t^2}{\partial t \partial r^2} - \frac{1}{c^2} (K^2)^{-1} \frac{\partial R \partial^2 R}{\partial r^2} = b_1
\] (9.13)
on \( r = r_+ \), where \( b_1 \) is a function of the values of \( \partial_t t^2, \partial_r t^2, \partial_r \partial_t t^2, R, \partial_r R, \partial_t \partial_r R \) on \( r = r_+ \). The continuity of \( \partial_t g_{11} \) reads a condition of the form

\[
c^2 K^2 \frac{\partial t^2}{\partial r \partial r^2} - (K^2)^{-1} \frac{\partial R \partial^2 R}{\partial r^2} = b_2,
\] (9.14)
on \( r = r_+ \), where \( b_2 \) is a function of the same kind as \( b_1 \). If we consider (9.13)(9.14) as a system of simultaneous linear equations for the unknown \( \partial^2 t^2 / \partial r^2, \partial^2 R^2 / \partial r^2 \), the determinant of the coefficient matrix is

\[-\sqrt{\kappa} \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} \right)^{-1/2} \frac{\partial R}{\partial r},\]

which is near to \(-1\), since

\[1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} \equiv \kappa, \quad \frac{\partial R}{\partial r} \equiv 1.\]

Since \( b_1, b_2 \) are known by (9.11)(9.12), the values \( \partial^2 t^2 / \partial r^2, \partial^2 R^2 / \partial r^2 \) along \( r = r_+ + 0 \) are uniquely determined. Then all \( g_{\mu \nu} \) are of class \( C^4 \).

However we note that this \( \partial^2 R^2 / \partial r^2 \) generally does not coincide with \( \partial^2 R / \partial r^2 \) on \( r = r_+ \), which coincidence is necessary to that \( g_{22} \) be twice continuously differentiable. In fact by a tedious calculation we get

\[
\left. \frac{\partial^2 R^2}{\partial r^2} \right|_{r=r_+} = A' \left( \frac{\partial R}{\partial r} \right)^2 + \left. \frac{\partial^2 R}{\partial r^2} \right|_{r=r_+} - \frac{1}{\kappa} \left( \frac{\partial V}{\partial r} \right)^2.
\]

where

\[
A = -\frac{V^2}{c^2} \left( \frac{Gm_+}{c^2 R^2} + \frac{1}{\sqrt{\kappa}} \frac{1}{c^2} \frac{\partial V}{\partial t} \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} \right)^{-2}
\]

evaluated at \( r = r_+ - 0 \). Since

\[
\frac{Gm_+}{c^2 R^2} \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} \right)^{-2} \equiv \frac{Gm_+}{c^2 r_+^2 \kappa^2} \neq 0,
\]

we see that \( \partial^2 R^2 / \partial r^2 \equiv \partial^2 R / \partial r^2 \) if and only if \( V \equiv 0 \) at \( r = r_+ - 0 \), which is the case if the solution under consideration is an equilibrium.

Anyway we have determined the functions

\[
f_0(t) := R(t, r_+), \quad f_1(t) := \partial_t R(t, r_+),
\]

\[
f_2(t) := \left. \frac{\partial^2 R^2}{\partial r^2} \right|_{r=r_+ + 0}, \quad H(t) := \left. \frac{\partial t^2}{\partial t} \right|_{r=R_+ + 0}, \quad h_0(t) := \int_0^t H(t') dt',
\]

\[
h_1(t) := \left. \frac{\partial t^2}{\partial r} \right|_{r=r_+ + 0}, \quad h_2(t) := \left. \frac{\partial^2 R^2}{\partial r^2} \right|_{r=r_+ + 0}
\]
for $0 \leq t \leq T$. Using these functions we define $\theta^2(t, r), R^2(t, r)$ for $0 \leq t \leq T, r_+ \leq r < +\infty$ as follow:

\[
\begin{align*}
R^2(t, r) &= f_0(t) + f_1(t)(r - r_+) + \frac{1}{2} f_2(t)(r - r_+)^2 \chi(r - r_+), \\
\theta^2(t, r) &= h_0(t) + \left( h_1(t)(r - r_+) + \frac{1}{2} h_2(t)(r - r_+)^2 \right) \chi(\delta(r - r_+)).
\end{align*}
\]

Here $\chi$ is a smooth cut off function in $C^\infty[0, +\infty)$ such that $0 \leq \chi(s) \leq 1, \chi(s) = 1$ for $0 \leq s \leq 1$ and $\chi(s) = 0$ for $2 \leq s < +\infty$ and $\delta$ is a sufficiently small positive number. Since $f_0(t) \equiv r_+, f_1(t) \equiv 1, f_2(t) \equiv 0, H(t) \equiv 1$, we see that $\partial R^2 / \partial r \equiv 1$ and $\partial \theta^2 / \partial t \equiv 1$ uniformly. Then the coefficients of the metric $g_{00}, g_{01}, g_{11}$ and $g_{22}$ are of class $C^1([0, T] \times [0, +\infty))$ and their second order derivatives may have discontinuity of at most the first kind along the segment $r = r_+$, and satisfy the Einstein equations in the usual sense on $r \neq r_+$. So, we can say that this metric is a weak solution of the Einstein equations on $[0, T] \times \mathbb{R}^3$ in the following sense: The Einstein equations can be written as

\[
R_{\mu \nu} = \frac{8\pi G}{c^4} \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right)
\]

and

\[
T = T^{\alpha \beta} T_{\alpha \beta},
\]

\[
R_{\mu \nu} = \frac{1}{2} g^{\alpha \beta} \left( -\partial_\alpha \partial_\beta g_{\mu \nu} - \partial_\mu \partial_\nu g_{\alpha \beta} + \partial_\alpha \partial_\nu g_{\mu \beta} + \partial_\mu \partial_\beta g_{\nu \alpha} \right) + F_{\mu \nu},
\]

\[
F_{\mu \nu} = \frac{1}{2} \partial_\alpha g^{\alpha \beta} \left( \partial_\nu g_{\beta \mu} + \partial_\mu g_{\beta \nu} - \partial_\beta g_{\mu \nu} \right) +
\]

\[
- \frac{1}{2} \partial_\alpha g^{\alpha \beta} \left( \partial_\nu g_{\beta \mu} + \partial_\mu g_{\beta \nu} - \partial_\beta g_{\mu \nu} \right);\]

Therefore, $(T_{\mu \nu})_{\mu \nu}, T \in L^2_{\text{loc}}$ given, $ds^2 = g_{\mu \nu} dx^\mu dx^\nu$ is said to be a weak solution if $g_{\mu \nu}, g^{\alpha \beta} \in H^1_{\text{loc}}$ and for any test function $(\phi^\mu)_{\mu \nu}$ there holds

\[
\frac{1}{2} \int \left( \partial_\beta g_{\mu \nu} \partial_\alpha (g^{\alpha \beta} \phi^\mu) + (\partial_\nu g_{\alpha \beta}) \partial_\mu (g^{\alpha \beta} \phi^\nu) +
\right.
\]

\[
- (\partial_\nu g_{\alpha \beta}) \partial_\beta (g^{\alpha \beta} \phi^\nu) - (\partial_{\alpha} g_{\beta \nu}) \partial_\mu (g^{\alpha \beta} \phi^\mu) +
\left.
\int F_{\mu \nu} \phi^\nu =
\right)
\]

\[
= \frac{8\pi G}{c^4} \int \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \phi^\nu.
\]

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**Appendix**

Let us give an outline of the tame estimate of the mapping $(\vec{w}, \vec{g}) \rightarrow \vec{h}$ when $D\mathfrak{F}(\vec{w})\vec{h} = \vec{g}$. The equation (8.4) is split as [9] using a cut off function $\omega \in C^\infty$ such that $\omega(x) = 1$ for $x \leq 1/3$, $0 < \omega(x) < 1$ for $1/3 < x < 2/3$ and $\omega(x) = 0$ for $2/3 \leq x$. Put

$$\vec{h}^{[0]}(x) = \omega(x)\vec{h}(x), \quad \vec{h}^{[1]}(x) = (1 - \omega(x))\vec{h}(x).$$

The equations turn out to be

$$\frac{\partial}{\partial t} \begin{bmatrix} h^{[\mu]} \\ k^{[\mu]} \end{bmatrix} + \begin{bmatrix} a_1^{[\mu]} - J \\ A^{[\mu]} \\ a_2^{[\mu]} \end{bmatrix} \begin{bmatrix} h^{[\mu]} \\ k^{[\mu]} \end{bmatrix} = \begin{bmatrix} g_1^{[\mu]} \\ g_2^{[\mu]} \end{bmatrix} + (-1)^\mu \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} h^{[1-\mu]} \\ k^{[1-\mu]} \end{bmatrix},$$

where $\mu = 0, 1$ and

- $a_1^{[\mu]} = a_{01}\bar{D} + a_{00} - (-1)^\mu a_{01}\bar{D}\omega$,
- $a_2^{[\mu]} = a_{21}\bar{D} + a_{20} - (-1)^\mu a_{21}\bar{D}\omega$,
- $A^{[\mu]} = -H_1\Lambda + (b_1 + (-1)^\mu 2H_1(D\omega))\bar{D} + b_0 + (-1)^\mu (H_1\Lambda - b_1\bar{D})\omega$,
- $c_{11} = a_{01}\bar{D}\omega$,
- $c_{21} = -2H_1(D\omega)\bar{D} + b(\bar{D}\omega) - H_1(\Lambda\omega)$,
- $c_{22} = a_{21}\bar{D}\omega$.

Therefore the problem is reduced to the tame estimate of an equation of the form

$$\frac{\partial \vec{h}}{\partial t} + \mathfrak{A}\vec{h} = \vec{g},$$

$$\mathfrak{A} = \begin{bmatrix} a_1 & J \\ A & a_2 \end{bmatrix} = \begin{bmatrix} a_{01}\bar{D} + a_{00} \\ -b_2\Delta + b_1\bar{D} + b_0 \end{bmatrix} \begin{bmatrix} a_{21}\bar{D} + a_{20} \\ J \end{bmatrix}$$

under the boundary condition $h|_{x=1} = 0$, where

$$\Delta = x \frac{d^2}{dx^2} + N \frac{d}{dx}, \quad \bar{D} = x \frac{d}{dx}$$

with $N$ standing for either $2\gamma/(\gamma - 1)$ or 5.
As in [9], we use the notations
\[ \vec{a} = (a_i)_{i=0}^7 = (b_0, b_1, a_{00}, a_{01}, a_{20}, a_{21}, J), \]
\[ |\vec{a}|_n^{(T)} = \sup_{0 \leq t \leq T} |\vec{a}|_n, \]
\[ |\vec{a}|_n = \max_{j+k \leq n, 0 \leq i \leq 7} \| \partial_t^j \dot{a}_i \|_{L^\infty} \]
\[ \| \vec{h} \|_n^{(T)} = \left( \sum_{j+k \leq n} \int_0^T \| \partial_t^j \dot{h} \|_2^2 dt \right)^{1/2} \]
\[ \| \vec{h} \|_k = \left( \sum_{0 \leq \ell \leq k} (\langle h \rangle_{\ell+1}^2 + \langle k \rangle_{\ell}^2) \right)^{1/2}. \]

Here \( \langle \phi \rangle_{\ell} \) means the same as [8].

Then the elliptic a priori estimate [8, Proposition 8] should read
\[ \| \vec{h} \|_{n+1} \leq C(\| 2\vec{h} \|_n + (1 + |\vec{a}|_{n+4}) \| h \|_0). \]

This can be verified if we keep in mind that
\[ \| a_1 h \|_1 \leq C(\| e \|_1 \| h \|_2 + \| h \|_1), \]
\[ \| a_2 k \|_0 \leq C(\| e \| \| k \|_1 + \| k \|_0), \]
which come from
\[ a_{01} = \frac{1}{c_2} e^F \frac{P \rho}{\rho} (1 + y + z)^{-1} \varepsilon(V_1 + V) \frac{r}{x(1 - x)} dx, \]
\[ a_{21} = -\frac{1}{c_2} e^F \frac{P \rho}{\rho} (1 + y)^2 \varepsilon(V_1 + V) \frac{r}{x(1 - x)} dx. \]

In fact estimates of the commutators
\[ \| [\triangle, \mathcal{A}] \phi \|_n \leq C(\| a_2 \|_2 \phi \|_{n+3} + |\vec{a}|_{n+5} \| \phi \|_0), \]
\[ \| [\triangle, \mathcal{A}_i] \phi \|_n \leq C(\| a_3 \|_i \phi \|_{n+2} + |\vec{a}|_{n+5} \| \phi \|_0), \]
\[ \| [\triangle, \mathcal{K}] \phi \|_n \leq C(\| a_4 \|_i \phi \|_{n+1} + |\vec{a}|_{n+5} \| \phi \|_0) \]
can be derived as in [8] and used to prove the elliptic a priori estimate by induction on \( n \).

On the other hand the energy estimate should read
\[ \| \vec{H} \| \leq C \left( \| \vec{H} \|_{t=0} \| + \int_0^T \| \vec{G}(t') \| dt' \right), \]
where
\[ \| \vec{H} \| = (\| H \|^2 + \| D H \|^2 + \| K \|^2)^{1/2} \] with \( \| \cdot \| = \| \cdot \| _{L^2(\mathbb{R}^n \times [0, T])}. \)
for any solution $\vec{H} = (H, K)^T$ of

$$\frac{\partial \vec{H}}{\partial t} + \mathfrak{A}\vec{H} = \vec{G}, \quad H|_{x=1} = 0,$$

which may not vanish at $t = 0$, so that

$$\|\partial^n_H\vec{h}\| \leq C\left(\|\partial^n_H\vec{h}\|_{t=0} + \int_0^t \|\partial^n_H\vec{g}\| + \int_0^t \|\partial^n_H, \mathfrak{A}\vec{h}\|\right).$$

Moreover we have an estimate

$$\|\partial^{n+1}_t \vec{h}\|_{t=0} \leq C(1 + W_n(\vec{g}) + |\vec{a}|^{(0)}_{n+3}),$$

where

$$W_n(\vec{g}) = \sum_{j+k \leq n} \|\partial^j_t \vec{g}\|_{t=0} \|k\|,$$

provided that $|\vec{a}|_4$ and $W_0(\vec{g})$ are bounded. In order to verify this, it is sufficient to show

$$\|\partial^{n+1}_t \vec{h}\|_{t=0} \leq C(W_{n+k}(\vec{g}) + |\vec{a}|^{(0)}_{n+k+3} W_0(\vec{g}) + |\vec{a}|^{(0)}_{k+4} W_{n-1}(\vec{g}))$$

inductively on $n$ using

$$\|\mathfrak{A}\vec{h}\|_n \leq C(\|\vec{h}\|_{n+1} + |\vec{a}|_{n+4} \|\vec{h}\|_0).$$

Then the same discussion using the auxiliary quantity

$$Z_n(\vec{h}) = \sum_{j+k = n} \|\partial^j_t \vec{h}\|_k$$

as [8] leads us to the estimate

$$\|\vec{h}\|_{n+1}^{(t)} \leq C\left(1 + \int_0^t \|\vec{g}\|_{n+1}^{(t')} dt' + W_n(\vec{g}) + \|\vec{g}\|_{n}^{(T)} + |\vec{a}|_{n+3}^{(T)}\right)$$

for $0 \leq t \leq T$.

This estimate for the split problem is sufficient to get the tame estimate for the original $h = h^{(0)} + h^{(1)}$ as [9]. We omit the repetition of the discussion.

References


