Equation of Linear Non-radial Oscillations of Gaseous Stars

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July 9, 2019
Equadiff 2019, at Leiden
Problem
Euler-Poisson equations on \((t, \mathbf{x}) = (t, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in [0, +\infty[ \times \mathbb{R}^3:\)

\[
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\rho v^k) = 0, \quad (1a)
\]

\[
\rho \left( \frac{\partial v^j}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial v^j}{\partial x^k} \right) + \frac{\partial P}{\partial x^j} + \rho \frac{\partial \Phi}{\partial x^j} = 0, \quad (j = 1, 2, 3), \quad (1b)
\]

\[
\rho \left( \frac{\partial S}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial S}{\partial x^k} \right) = 0, \quad (1c)
\]

\[
\Delta \Phi = 4\pi G \rho. \quad (1d)
\]
Supposing that the support of \( \rho(t, \cdot) \) is compact, we replace the Poisson equation (1d) by the Newtonian potential

\[
\Phi = -4\pi G K \rho(t, \cdot),
\]

(2)

where the integral operator \( K \) is defined as

\[
K f(x) = \frac{1}{4\pi} \int \frac{f(x')}{\|x - x'\|} dx'.
\]

(3)
Assumption 1.

\[ P = \rho^\gamma \exp \left( \frac{S}{C_V} \right) \]  \hspace{1cm} (4)

for \( \rho > 0 \), where \( \gamma, C_V \) are positive constants such that \( 1 < \gamma < 2 \).
Definition 1. admissible spherically symmetric equilibrium
\((\bar{\rho}, \bar{S})\) a function of \(r = \|x\| : \bar{\rho} \in C^1_0(\mathbb{R}^3, [0, +\infty[), \bar{S} \in C^1(\mathbb{R}^3, \mathbb{R}), \rho = \bar{\rho}, S = \bar{S}, v = \mathbf{0}\) satisfies (1)(2), \(\exists R < \infty\) such that
1) \(\{\bar{\rho} > 0\} = B_R(:= \{x \in \mathbb{R}^3 \mid r = \|x\| < R\}),\)
2) \(\bar{\rho}^{\gamma-1}, \bar{S} \in C^\infty(B_R) \cap C^{2,\alpha}(\overline{B_R})\) with \(0 < \exists \alpha < \left(\frac{\gamma}{\gamma-1} - 2\right) \wedge 1,\)
3) \(d\bar{\rho}/dr, d\bar{P}/dr < 0\) for \(0 < r < R,\)
\[\exists \lim_{r \to +0} \frac{1}{r} \frac{d\bar{\rho}}{dr} =: -\rho_{O1} < 0, \quad \exists \lim_{r \to +0} \frac{1}{r} \frac{d\bar{P}}{dr} =: -P_{O1} < 0.\]
4) 
\[-\infty < \frac{d}{dr} \bar{\rho}^{\gamma-1}\bigg|_{r=R-0} < 0.\]
Assumption 2. \( \exists \) admissible spherically symmetric equilibrium \((\bar{\rho}, \bar{S})\).

Fix one of the admissible spherically symmetric equilibria.

We consider small perturbation from this fixed equilibrium by the Lagrangian co-ordinate system, which will be denoted by the same letters \((t, x^1, x^2, x^3)\) of the Eulerian co-ordinate system.

It is known that, when the initial perturbations \(\rho - \bar{\rho}, S - \bar{S}\) at \(t = 0\) vanish, the linearized approximation of the equations for the perturbations \(\xi = \sum \xi^k \partial / \partial x^k\) of \(x\) is

\[
\frac{\partial^2 \xi}{\partial t^2} + L\xi = 0,
\]

(5)
where

\[ L\xi = \frac{1}{\bar{\rho}} \text{grad}\delta P - \frac{\text{grad}P}{\rho^2} \delta \rho + \text{grad}\delta \Phi, \quad (6) \]

\[ \delta \rho = -\text{div}(\bar{\rho}\xi), \quad (7) \]

\[ \delta \Phi = -4\pi G \mathcal{K}(\delta \rho), \quad (8) \]

\[ \delta P = \frac{\gamma P}{\rho} \delta \rho + \gamma \mathcal{A} \bar{P}(\xi|e_r). \quad (9) \]

Here

**Definition 2.**

\[ \mathcal{A} := \frac{1}{\rho} \frac{d\rho}{dr} - \frac{1}{\gamma P} \frac{dP}{dr} = -\frac{1}{\gamma C_V} \frac{dS}{dr}, \quad (10a) \]

\[ \mathcal{N}^2 := \mathcal{A} \frac{1}{\rho} \frac{dP}{dr} = -\mathcal{A} \frac{d\Phi}{dr}. \quad (10b) \]
Problem: Clarify the spectral properties of the operator $L$ in the Hilbert space $\mathcal{H} := L^2(B_R, \bar{\rho} dx)$. 
Part I : Isentropic case

Joint work with Juhi Jang (University of Southern California, Korea Institute for Advanced Study)

arXiv: 1810.08294
Suppose $S = \text{Constant}$.

\[ P = A\rho^\gamma, \quad A := \exp\left(\frac{S}{C_V}\right) = \text{Const.} \]

\[ \mathcal{A} = -\frac{1}{\gamma C_V} \frac{d\bar{S}}{dr} = 0. \]

Put

\[ u := \int_0^\rho \frac{dP}{\rho} = \frac{A\gamma}{\gamma - 1} \rho^{\gamma - 1}, \tag{11} \]
If $\frac{6}{5} < \gamma < 2$, then $\exists$ admissible equilibrium for $\forall \rho_0 > 0$:

$$\rho = \bar{\rho} = \rho_0 \theta \left( \frac{r}{a}; \nu \right)^\nu, \quad u = \bar{u} = u_0 \theta \left( \frac{r}{a}; \nu \right)$$  \hspace{1cm} (12)

where $\nu := \frac{1}{\gamma - 1}$, $a = \sqrt[4]{\frac{A \gamma}{4\pi G(\gamma - 1)}} \rho_0^{\gamma - 2}$ and $\theta(\xi; \nu)$ is the Lane-Emden equation of index $\nu$:

$$- \frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \frac{d\theta}{d\xi} = (\theta \vee 0)^\nu, \quad \theta = 1 + O(\xi^2) \quad \text{as} \quad \xi = +0$$

$$\theta(\xi_1(\nu)) = 0, \quad \mu_1(\nu) = -\xi^2 \left. \frac{d\theta}{d\xi} \right|_{\xi = \xi_1(\nu)} < 0.$$
We are considering

$$\frac{\partial^2 \xi}{\partial t^2} + L\xi = 0,$$

(13)

with

$$L\xi = \text{grad} \left( -\frac{1}{\rho} \frac{dP}{d\rho} \text{div}(\bar{\rho}\xi) + 4\pi G\kappa \text{div}(\bar{\rho}\xi) \right),$$

(14)

thanks to $A = -\frac{1}{\gamma C_V} \frac{d\tilde{S}}{dr} = 0$.

We consider the integro-differential operator $L$ acting on the field $\xi$ in the Hilbert space $\mathcal{H} = L^2(B_R, \bar{\rho}dx)$
The operator $L \upharpoonright C_0^\infty(B_R) : \xi \in C_0^\infty(B_R; \mathbb{C}^3) \mapsto L\xi$ is symmetric, and is bounded from below.

Therefore the operator $L \upharpoonright C_0^\infty(B_R)$ admits the Friedrichs extension which is a self-adjoint operator in the Hilbert space $\mathcal{H}$.

Hereafter we shall denote by the same letter $L$ the Friedrichs extension. Thus we can claim the following

**Theorem 1.** The operator $L$ is a self-adjoint operator bounded from below in the Hilbert space $\mathcal{H}$. 
Definition 3. The spectrum $\sigma(T)$ of a self-adjoint operator $T$ in an infinitely dimensional Hilbert space $X$ is said to be of the Sturm-Liouville type if $\sigma(T)$ consists of eigenvalues with finite multiplicities which accumulate to $+\infty$.

The Riesz-Schauder’s theorem: If a resolvent $(\lambda - T)^{-1}$ of the self-adjoint operator $T$ bounded from below is a compact operator, then the spectrum of the operator $T$ is of the Sturm-Liouville type.
Theorem 2. *The spectrum of \( L \) is not of the Sturm-Liouville type.*

In fact, the functional space

\[
\mathcal{N} = \{ \xi \in \mathfrak{H} \mid \text{div}(\bar{\rho}\xi) = 0 \text{ in distribution sense} \} \quad (15)
\]

is a subset of Ker\( L \), that is, any \( \xi \in \mathcal{N}, \neq 0 \) is an eigenfunction of \( L \) associated with the eigenvalue 0, while the dimension of \( \mathcal{N} \) is infinite, since for any vector function \( A \in C_0^\infty(B_R) \), the vector function

\[
\xi = \frac{1}{\bar{\rho}} \text{curl} A
\]

belongs to \( \mathcal{N} \). \( \square \)
Taking the divergence of (13), we see that the equation for
\[ g(= -\delta \rho) = \text{div}(\bar{\rho} \xi) \] (16)
turns out to be
\[ \frac{\partial^2 g}{\partial t^2} + \mathcal{N} g = 0, \] (17)
where
\[ \mathcal{N} g = -\rho \frac{d\rho}{dP} \text{div}\left( \frac{1}{\rho} \left( \frac{dP}{d\rho} \right)^2 \text{grad} g \right) - \left[ \Delta \left( \frac{dP}{d\rho} - u \right) \right] g + 4\pi G \text{div}\left( \bar{\rho} \text{grad}(\mathcal{K} g) \right). \] (18)

Let us consider the operator $\mathcal{N}$ in a Hilbert space
\[ \mathfrak{S} = L^2\left( B_R; \frac{1}{\rho} \frac{dP}{d\rho} d\mathbf{x} \right) \cap \{ g | \int_{B_R} g d\mathbf{x} = 0 \}. \]
Using the theory of the weighted Sobolev spaces developed by the Czech school, we can claim

**Theorem 3.** The operator $\mathcal{N}$ is a self-adjoint operator bounded from below in the Hilbert space $\mathfrak{H}$ and its spectrum $\sigma(\mathcal{N})$ is of the Sturm-Liouville type.
Theorem 4. The operator $L$ is a self-adjoint operator in $\mathcal{F}$. Its spectrum $\sigma(L)$ coincides with $\sigma(N) \cup \{0\}$, while $\dim \ker(L) = \infty$ and $\lambda \in \sigma(L) \setminus \{0\}$ is an eigenvalue of finite multiplicity.

Here $\mathcal{F}$ is the Hilbert space endowed with the inner product

$$(\xi_1|\xi_2)_{\mathcal{F}} = (\xi_1|\xi_2)_{\mathcal{H}} + (\text{div}(\rho \xi_1)|\text{div}(\rho \xi_2))_{\mathcal{G}}.$$
Remark 1. Here $L$ stands for the Friedrichs extension of $L|_{C^\infty_0(B_R)}$ in $\mathcal{F}$, and is different from the Friedrichs extension of $L|_{C^\infty_0(B_R)}$ in $\mathcal{H}$.

Open Problem 1. How about the spectrum $\sigma(L)$ of the operator $L$ considered in $\mathcal{H} = L^2(B_R, \bar{\rho}dx)$? For $\lambda \neq 0, \lambda \notin \sigma(N)$ can we claim that $(\lambda - L)^{-1}$ is bounded re $\mathcal{H}$-norm so that $\lambda \in \rho(L)$?
Part II: General Adiabatic Case

arXiv: 1902.03675
1 Existence of admissible spherically symmetric equilibrium

Theorem 5. Let \( \Sigma : \eta \mapsto \Sigma(\eta) \) which belongs to \( C^\infty(\mathbb{R}) \) and \( \rho_0 > 0 \) be given. Assume

\[
\gamma + \frac{\gamma - 1}{C_V} \eta \frac{d\Sigma}{d\eta} > 0 \quad \text{for} \quad \eta > 0.
\]

If \( \frac{4}{3} < \gamma < 2 \) or if \( \frac{6}{5} < \gamma \leq \frac{4}{3} \) and \( \rho_0 \ll 1 \), then \( \exists \) admissible spherically symmetric equilibrium \((\bar{\rho}, \bar{S})\) such that \( \bar{S} = \Sigma(\bar{\rho}^{\gamma-1}) \), \( \bar{\rho}(O) = \rho_0 \).
Proposition 1. Let $\bar{S} = \Sigma(\bar{\rho}^{\gamma-1})$. If

$$\frac{d\Sigma}{d\eta} < 0 \quad \text{for} \quad \eta > 0,$$

then $A < 0$ for $0 < r \leq R$. 
2 Self-adjoint realization of $L$

\[ L\xi = \frac{1}{\rho} \text{grad}\delta P - \frac{\text{grad}P}{\rho^2} \delta\rho + \text{grad}\delta\Phi, \]
\[ \delta\rho = -\text{div}(\bar{\rho}\xi), \]
\[ \delta P = \frac{\gamma P}{\rho} \delta\rho + \gamma A\bar{P}(\xi|e_r), \]
\[ \delta\Phi = -4\pi G\mathcal{K}(\delta\rho). \]

**Theorem 6.** $L \upharpoonright C_0^\infty(B_R)$ admits the Friedrichs extension, which is a self-adjoint operator bounded from below, in the Hilbert space $\mathcal{H} = L^2(B_R, \bar{\rho}dx)$. 

25
• $\dim \ker(L) = \infty$.

**Open Problem 2.** Let $A < 0$ everywhere on $0 < r < R$. Maybe it does not hold that

$$\sigma(L) = \{0\} \cup \{\lambda_n | n \in \mathbb{N}\}, \quad \dim \ker(\lambda_n - L) < \infty,$$

$$\lambda_n \to +\infty \quad (n \to \infty),$$

even if we restrict $L$ to some suitable $\mathcal{F} \subset \mathcal{H}$. Maybe there exist a sequence of eigenvalues $\lambda_{-n}$ such that $\lambda_{-n} \to 0$ as $n \to \infty$. (‘g-modes’ discussed below).
3 Eigenfunctions represented by spherical harmonics

\[ \xi = V^r(r)Y_{lm}(\vartheta, \phi)e_r + V^h(r)\nabla_s Y_{lm}(\vartheta, \phi). \] (19)

Here \( l, m \in \mathbb{Z}, 0 \leq l, |m| \leq l \), and \( Y_{lm} \) is the spherical harmonics:

\[
Y_{lm}(\vartheta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P^m_l(\cos \vartheta) e^{\sqrt{-1}m\phi},
\]

\[
Y_{l,-m} = (-1)^m Y^*_{lm} \quad \text{for} \quad m \geq 0.
\]

27
\[ x^1 = r \sin \vartheta \cos \phi, \]
\[ x^2 = r \sin \vartheta \sin \phi, \]
\[ x^3 = r \cos \vartheta. \]

\[ \nabla_s f := \frac{\partial f}{\partial \vartheta} e_\vartheta + \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \phi} e_\phi \]

\[ \delta \rho = \delta \tilde{\rho}(r) Y_{lm}(\vartheta, \phi), \quad (20a) \]
\[ \delta P = \delta \tilde{P}(r) Y_{lm}(\vartheta, \phi), \quad (20b) \]
\[ \delta \Phi = \delta \tilde{\Phi}(r) Y_{lm}(\vartheta, \phi), \quad (20c) \]
where

$$\delta \tilde{\rho} = -\frac{1}{r^2} \frac{d}{dr} (r^2 \rho V^r) + \frac{l(l+1)}{r} \rho V^h,$$  \hspace{1cm} (21a)

$$\delta \tilde{P} = \frac{\gamma P}{\rho} \delta \tilde{\rho} + \gamma A P V^r$$

$$= -\frac{\gamma P}{r^2 \rho} \frac{d}{dr} (r^2 \rho V^r) + \gamma A P V^r + l(l+1) \frac{\gamma P}{r} V^h,$$  \hspace{1cm} (21b)

$$\delta \tilde{\Phi} = -4\pi G \mathcal{H}_l(\delta \tilde{\rho})$$

$$= 4\pi G \mathcal{H}_l \left( \frac{1}{r^2} \frac{d}{dr} (r^2 \rho V^r) - \frac{l(l+1)}{r} \rho V^h \right).$$  \hspace{1cm} (21c)
Here the integral operator $\mathcal{H}_l$, which solves

$$
\left[ - \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{l(l+1)}{r^2} \right] \mathcal{H}_l f = f,
$$

is defined by

$$
\mathcal{H}_l f(r) = \frac{1}{2l+1} \left[ \int_r^\infty f(r') \left( \frac{r}{r'} \right)^l r' dr' + \int_0^r f(r') \left( \frac{r}{r'} \right)^{-(l+1)} r' dr' \right],
$$

(22)
We mean
\[ L(V^r Y_{lm} e_r + V^h \nabla_s Y_{lm}) = L^r_l Y_{lm} e_r + L^h_l \nabla_s Y_{lm}. \]  
(23)

Note \( Y_{00} = \frac{1}{\sqrt{4\pi}} \) so that we can forget \( L^h_0 \) and the component \( V^h \) for \( l = 0 \).

The wave equation reads
\[ \frac{\partial^2 V^r}{\partial t^2} + L^r_l = 0, \quad \frac{\partial^2 V^h}{\partial t^2} + L^h_l = 0, \]  
(24)
where

\[
L^r_l = \frac{1}{\rho} \frac{d}{dr} \delta \ddot{P} - \frac{1}{\rho^2} \frac{dP}{dr} \delta \dot{\rho} + \frac{d}{dr} \delta \ddot{\Phi},
\]

\[L^h_l = \frac{1}{r} \left( \frac{\delta \ddot{P}}{\rho} + \delta \ddot{\Phi} \right).\]

(25a) 

(25b)

We are going to analyze the operator \( \vec{L}_l = \begin{bmatrix} L^r_l & L^h_l \end{bmatrix} \) which acts on

\[
\vec{V} = \begin{bmatrix} V^r \\ V^h \end{bmatrix}.
\]
3.1 Case $l = 0$

**Theorem 7.** The operator $L^{ss}$ on $C_0^\infty([0, R])$ admits the Friedrichs extension, a self-adjoint operator bounded from below in $L^2([0, R], \bar{\rho}r^4\,dr)$, and its spectrum consists of simple eigenvalues $\lambda_1^{ss} < \lambda_2^{ss} < \cdots < \lambda_n^{ss} < \cdots \to +\infty$.

Here

$$L^{ss}\psi = \frac{1}{r}L^r_0((r\psi, 0)^\top),$$

while $L^h_0$ need not be considered.
3.2 Case $l \geq 1$

Suppose $l \geq 1$.

Let us consider the Hilbert space $\mathfrak{X}_l$ of functions $\vec{f} = (f^r, f^h)$ defined on $[0, R]$ endowed with the norm $\|\vec{f}\|_{\mathfrak{X}_l}$ given by

$$\|\vec{f}\|_{\mathfrak{X}_l}^2 = \int_0^R \left( |f^r(r)|^2 + l(l + 1)|f^h(r)|^2 \right) \rho r^2 dr. \quad (26)$$

Note

$$\|\xi\|_{\mathfrak{S}} = \sqrt{4\pi} \|\vec{V}\|_{\mathfrak{X}_l}$$

under (19).
Theorem 8. The integro-differential operator $\vec{L}_l$ on $C_0^\infty([0, R[, \mathbb{C}^2)$ admits the Friedrichs extension, which is a self-adjoint operator bounded from below, in $\mathfrak{X}_l$.

Note

$$(\vec{L}_l\vec{V}|\vec{V})_{\mathfrak{X}_l} = \gamma \int |W|^2 dr + \int \mathcal{A} \frac{dP}{dr} |V^r|^2 r^2 dr - 4\pi G \int \mathcal{H}_l(\delta \tilde{\rho})(\delta \tilde{\rho})^* r^2 dr,$$

where

$$W := \frac{P^{\frac{1}{2} - \frac{1}{\gamma}} d}{r} (r^2 P^{\frac{1}{2}} V^r) - l(l + 1) P^{\frac{1}{2}} V^h$$

$$= -\frac{1}{\gamma} r P^{-\frac{1}{2}} \delta \tilde{P}.$$
Note

\[ D(\tilde{L}_{l}) = \tilde{\mathcal{M}}_{l} \cap \{ \tilde{V} \mid \tilde{L}_{l}\tilde{V} \in \mathcal{X}_{l} \text{ in distribution sense} \}. \quad (27) \]

Here $\tilde{\mathcal{M}}_{l}$ is the closure of $C_{0}^{\infty}([0, R[, \mathbb{C}^{2})$ in the Hilbert space $\mathcal{M}_{l}$ endowed with the norm $\| \cdot \|_{\mathcal{M}_{l}}$ given by

\[ \| \tilde{V} \|_{\mathcal{M}_{l}}^{2} = \| \tilde{V} \|_{\mathcal{X}_{l}}^{2} + \| \delta \tilde{\rho} \|_{L^{2}(\frac{\gamma P}{\rho^{2}} r^{2} dr)}^{2}, \quad (28) \]

where

\[ \| \delta \tilde{\rho} \|_{L^{2}(\frac{\gamma P}{\rho^{2}} r^{2} dr)}^{2} = \int_{0}^{R} \left[ -\frac{1}{r^{2}} \frac{d}{dr}(r^{2} \rho V^{r}) + \frac{l(l+1)}{r} \rho V^{h} \right]^{2} \frac{\gamma P}{\rho^{2}} r^{2} dr. \quad (29) \]

**Proposition 2.** If $\tilde{V} \in \mathcal{M}_{l}$ satisfies $|V^{r}| \leq C$, then $\tilde{V}$ belongs to $\tilde{\mathcal{M}}_{l}$.
Theorem 9. 1) Let \( l \geq 1 \). Suppose that \( A = 0 \) identically on \( ]0, R[ \). Then \( \dim \ker(\vec{L}_l) = \infty \).

2) Suppose that \( A < 0 \) everywhere on \( ]0, R[ \). Then \( \dim \ker(\vec{L}_l) = 0 \) when \( l \geq 2 \) and \( \dim \ker(\vec{L}_1) = 1 \) when \( l = 1 \).

Open Problem 3. Let \( A < 0 \) everywhere on \( 0 < r < R \). Then maybe it does not hold that

\[
\sigma(\vec{L}_l) = \{0\} \cup \{\lambda_n^{[l]} \mid n \in \mathbb{N}\}, \quad \dim \ker(\lambda_n^{[l]} - \vec{L}_l) < \infty,
\]

\[
\lambda_n^{[l]} \to +\infty \quad (n \to \infty)
\]

which is the case if \( A = -\frac{1}{\gamma C_V} \frac{d\tilde{S}}{dr} = 0 \).
4 Cowling approximation, g-modes, p-modes

Let us consider the case in which \( l \geq 1, \lambda \neq 0 \).

The Cowling approximation:

\[
\vec{L}_0l \vec{V} = \lambda \vec{V},
\]

where

\[
\vec{L}_0l = \begin{bmatrix} L^r_{0l} \\ L^h_{0l} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} \frac{d}{dr} \delta \tilde{P} - \frac{1}{\rho^2} \frac{dP}{dr} \delta \tilde{\rho} \\ \frac{1}{r \rho} \delta \tilde{P} \end{bmatrix}.
\]
Introduce the variables

\[ v = r^2 P^{\frac{1}{\gamma}} V^r, \quad w = P^{-\frac{1}{\gamma}} \delta \tilde{P}. \] (32)

by which

\[ V^r = \frac{1}{r^2} P^{-\frac{1}{\gamma}} v, \] (33a)

\[ V^h = \frac{1}{l(l + 1)} \left[ \frac{1}{r} P^{-\frac{1}{\gamma}} \frac{dv}{dr} + \frac{1}{\gamma} r P^{-1 + \frac{1}{\gamma}} w \right], \] (33b)

\[ \delta \tilde{P} = P^{\frac{1}{\gamma}} w, \] (33c)

\[ \delta \tilde{\rho} = \frac{1}{\gamma} \rho P^{-1 + \frac{1}{\gamma}} w - \frac{1}{r^2} A \rho P^{-\frac{1}{\gamma}} v, \] (33d)

\[ \frac{d}{dr} \delta \tilde{P} - \frac{1}{\rho} \frac{dP}{dr} \delta \tilde{\rho} = P^{\frac{1}{\gamma}} \frac{dw}{dr} + \frac{1}{r^2} A P^{-\frac{1}{\gamma}} \frac{dP}{dr} v. \] (33e)
The eigenvalue problem (30) reads

\[
\frac{dv}{dr} + \frac{r^2 \rho}{\gamma P} \mathcal{B} w = \frac{1}{\lambda} l(l + 1) \mathcal{B} w, \tag{34a}
\]

\[
\frac{dw}{dr} + \frac{N^2}{r^2 \mathcal{B}} v = \lambda \frac{1}{r^2 \mathcal{B}} v. \tag{34b}
\]

Here

\[
\mathcal{B} := \frac{P_{\gamma}^2}{\rho}, \quad N^2 = \frac{A}{\rho} \frac{dP}{dr}. \tag{35}
\]
**Assumption 3.** There is a positive number $C$ such that it holds that

$$\frac{1}{C^r} \leq \frac{d\tilde{S}}{dr}$$

for $0 < r \leq R$. 

(36)
[g-modes]

Put $\lambda = 0$ in (34b):

$$\frac{dw}{dr} + \frac{N^2}{r^2 B} v = 0.$$ 

Then (34a) turns to be

$$L^\text{[g]}_l w = \frac{1}{\lambda} w, \quad (37)$$

where

$$L^\text{[g]}_l w = -\frac{1}{l(l+1)B} \frac{d}{dr} \left( \frac{r^2 B}{N^2} \frac{dw}{dr} \right) + \frac{r^2 \rho}{l(l+1)\gamma P} w. \quad (38)$$
Theorem 10. The operator $L_{l}^{[g]}$ defined on $C_{0}^{\infty}(]0, R[)$ admits the Friedrichs extension, a self-adjoint operator in $L^{2}([0, R], l(l + 1)Bdr)$, whose spectrum consists of simple or double eigenvalues $\frac{1}{\lambda_{n}}, n = 1, 2, \cdots$ : $\frac{1}{\lambda_{1}^{g}} \leq \frac{1}{\lambda_{2}^{g}} \leq \cdots \leq \frac{1}{\lambda_{n}^{g}} \leq \rightarrow +\infty$. The eigenvalues are simple if $\gamma > 3/2$. 
[p-modes]

Put $\lambda = \infty$ at (34a). Then (34b) turns out to be

$$L_l^{[p]} v = \lambda v, \quad L_l^{[p]} v = -r^2 B \frac{d}{dr} \left( \frac{\gamma P}{r^2 \rho B} \frac{dv}{dr} \right) + N^2 v.$$ 

**Theorem 11.** The operator $L_l^{[p]}$ defined on $C_0^\infty(\mathbb{R})$ admits the Friedrichs extension, a self-adjoint operator in $L^2([0, R], \frac{1}{r^2 B} dr)$, whose spectrum consists of simple eigenvalues $\lambda_n^p, n = 1, 2, \cdots$, : $\lambda_1^p < \lambda_2^p < \cdots < \lambda_n^p < \cdots \to +\infty$. 
Open Problem 4. Let $l \geq 1$. We have had two sequences of eigenvalues $\lambda_{-n}^{[l]} = \lambda_n^g, \lambda_n^{[l]} = \lambda_n^p$, such that $\lambda_{\nu}^{[l]}$ tends to $+0$ as $\nu \to -\infty$ and to $+\infty$ as $\nu \to +\infty$. Can they give a good approximation for the eigenvalues of $\vec{L}_{0l}$?

Maybe justification of the Cowling approximation should be relatively easy.
THANK YOU
FOR YOUR ATTENTION!

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(http://hc3.seikyou.ne.jp/home/Tetu.Makino)