A Note on Birkhoff’s Theorem

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June 14, 2015

The standard textbook Misner et al [1] says:

That the Schwarzschild geometry is relevant to gravitational collapse follows from Birkhoff’s (1923) theorem: Let the geometry of a given region of spacetime (1) be spherically symmetric, and (2) be a solution to the Einstein field equations in vacuum. Then that geometry is necessarily a piece of the Schwarzschild geometry. The external field of any electrically neutral, spherical star satisfies the conditions of Birkhoff’s theorem, whether the star is static, vibrating, or collapsing. Therefore the external field must be a piece of the Schwarzschild geometry.

Just as Maxwell’s laws prohibit monopole electrodynamic waves, so Einstein’s laws prohibit monopole gravitational waves. There is no possible way for any gravitational influence of the radial collapse to propagate outward.

([1, p.843])

Although [1] continuing says ‘Not only is Birkhoff’s theorem easy to understand, but it is also fairly easy to prove’, mathematically rigorous proof seems not so easy. This note is devoted to a trial of mathematically rigorous formulation and proof of Birkhoff’s theorem.

Theorem 1 Suppose the metric

\[ ds^2 = \alpha(t, r)dt^2 + 2\delta(t, r)dt dr + \gamma(t, r)dr^2 + \beta(t, r)d\Omega^2, \]
\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \]

satisfies the vacuum Einstein equations \( G_{\mu\nu} = 0 \) on the domain

\[ \mathcal{D} = \{(t, r) \mid t_1 < t < t_2, \ r_1 < r < r_2\}, \]
\[ -\infty \leq t_1 < t_2 \leq +\infty, \ 0 \leq r_1 < r_2 \leq +\infty. \]

Here we assume that the coefficients \( \alpha, \beta, \gamma, \delta \) are \( C^2 \)-function of \((t, r) \in \mathcal{D}. \)

1) If \( \beta = -r^2 \) and \( \delta = 0, \) then there is a constant \( C \) such that

\[ ds^2 = \left(1 - \frac{C}{r}\right)^2 dt^2 - \left(1 - \frac{C}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \]
where $T = T(t)$ is a monotone increasing $C^2$-function of $t \in (t_1, t_2)$. 

2) Let $(t_0, r_0)$ be a point of $D$ at which $\partial \beta / \partial r \neq 0$ and

\[
\alpha \left( \frac{\partial \beta}{\partial r} \right)^2 - 2\delta \frac{\partial \beta}{\partial t} \frac{\partial \beta}{\partial r} + \gamma \left( \frac{\partial \beta}{\partial t} \right)^2 \neq 0.
\]

(1)

Then there is a $C^2$-diffeomorphism $\Phi: (t, r) \mapsto (T, R)$ defined on a neighborhood $U$ of $(t_0, r_0)$ such that

\[
ds^2 = A(T, R) dT^2 + \Gamma(T, R) dR^2 - R^2 d\Omega^2.
\]

Here $A, \Gamma$ are $C^2$-functions of $(T, R)$.

Proof of 1). We are considering the metric

\[
ds^2 = \alpha(t, r) dt^2 + \gamma(t, r) dr^2 - r^2 d\Omega^2.
\]

Since the eigenvalues of the matrix of the coefficients of the metric are $\alpha, \gamma, -r^2, -r^2 \sin^2 \theta$, we know $\alpha \cdot \gamma < 0$. Thus we can define $F, H$ such that $\alpha = e^{2H}, \gamma = -e^{2H}$. ($F, H$ can be purely imaginary.) Then $G_1^0 = 0$ implies

\[
\dot{H} = 0,
\]

(\(\dot{H}\) stands for $\partial H / \partial t$), which says that $H = H(r)$ is a function of only $r$. On the other hand $G_1^0 = 0$ implies

\[
e^{-2H} \left( \frac{1}{r^2} + \frac{2}{r} \frac{H'}{r} \right) + \frac{1}{r^2} = 0
\]

(3) and $G_0^0 = 0$ implies

\[
e^{-2H} \left( - \frac{1}{r^2} + \frac{2}{r} \frac{F'}{r} \right) + \frac{1}{r^2} = 0.
\]

(4) (Here $H', F'$ stand for $\partial H / \partial r, \partial F / \partial r$.) Then (3) − (4) gives $H' + F' = 0$, that is, $H + F = f(t)$ is a function of only $t$. Introducing the new variable $T$ by

\[
T(t) = \int^t \exp \left[ \frac{1}{2} f(t') \right] dt'
\]

instead of $t$, we can assume that $F = -H$. Solving the ordinary differential equation (3), we get

\[
e^{-2H(r)} = 1 - \frac{C}{r}
\]

with an arbitrary constant $C$. □

Proof of 2). We are considering a metric of the form

\[
ds^2 = \alpha dt^2 + 2\delta dt dr + \gamma dr^2 + \beta d\Omega^2.
\]

Let $\lambda_1, \lambda_2$ be the eigenvalues of the matrix

\[
\begin{bmatrix}
\alpha & \delta \\
\delta & \gamma
\end{bmatrix}
\]
Then the eigenvalues of the matrix of the coefficients of the metric are 
\( \lambda_1, \lambda_2, \beta, \beta \sin^2 \theta \). Since the signature should be \((+, -, -)\), we see \( \beta < 0 \). Hence we can define \( R(t, r) := \sqrt{-\beta(t, r)} \). Thanks to \( \partial \beta / \partial r \neq 0 \), we can assume that \( \partial R / \partial r > 0 \) or \( \partial R / \partial r < 0 \) on a neighborhood \( U \) of \((t_0, r_0)\). Then \((t, r) \mapsto (t, R)\) is a local \( C^2 \)-diffeo, and

\[
\begin{align*}
    ds^2 = \hat{\alpha} dt^2 + \hat{\delta} dt dR + \tilde{\gamma} dR^2 - R^2 d\Omega^2,
\end{align*}
\]

where

\[
\begin{align*}
    \hat{\alpha} &= \alpha + 2\delta X + \gamma X^2, \\
    \hat{\delta} &= (\delta + \gamma X) \frac{1}{\partial R / \partial r}, \\
    \tilde{\gamma} &= \gamma \left( \frac{1}{\partial R / \partial r} \right)^2.
\end{align*}
\]

Here

\[
X = -\frac{\partial \beta / \partial t}{\partial \beta / \partial r} = -\frac{\partial R / \partial t}{\partial R / \partial r}.
\]

Thanks to (1) we can assume that \( \hat{\alpha} \neq 0 \) near \((t_0, R_0)\), where \( R_0 = R(t_0, r_0) \).

Next we take a change of variable \((t, R) \mapsto (T, R)\) with \( T = T(t, R) \). Then

\[
\begin{align*}
    ds^2 = A dT^2 + 2\Delta dT dR + \Gamma dR^2 - R^2 d\Omega^2,
\end{align*}
\]

where

\[
\begin{align*}
    A &= \hat{\alpha} \left( \frac{\partial t}{\partial T} \right)^2, \\
    \Delta &= \left( \hat{\alpha} \frac{\partial t}{\partial R} + \hat{\delta} \right) \frac{\partial t}{\partial T}, \\
    \Gamma &= \hat{\alpha} \left( \frac{\partial t}{\partial R} \right)^2 + 2\hat{\delta} \frac{\partial t}{\partial R} + \tilde{\gamma}.
\end{align*}
\]

We want to find \( T(t, R) \) for which the inverse function \( t(T, R) \) satisfy

\[
\hat{\alpha} \frac{\partial t}{\partial R} + \hat{\delta} = 0,
\]

which implies \( \Delta = 0 \). Recall that \( \hat{\alpha} \neq 0 \) near \((t_0, R_0)\). The above condition can be considered as an ordinary differential equation with respect to the unknown function \( R \mapsto t(T, R) \) with parameter \( T \), say,

\[
\begin{align*}
    \frac{d}{dR} t(T, R) = -\frac{\hat{\delta}(T, R, R)}{\hat{\alpha}(T, R, R)}.
\end{align*}
\]

This ODE can be uniquely solved under the initial condition

\[
    t(T, R_0) = T.
\]
This introduce the mapping \((T, R) \mapsto (t, R)\) defined on a neighborhood of \((T_0(:= t_0), R_0)\), which is a local \(C^2\)-diffeo. Here we note that

\[
\frac{\partial}{\partial T} t(T, R_0) = 1
\]

so that \(\partial t/\partial T > 0\) on a neighborhood of \((T_0, R_0)\). □

B. F. Schutz says: ‘One conclusion one can draw from this (= Birkhoff’s theorem described in [1]) is that there are no gravitational waves from pulsating spherical systems.’ ([2, p. 258].) But if one wants draw this conclusion, he/she should first prove that the outer metric satisfies the assumptions of Theorem. It seems not so trivial.

References
